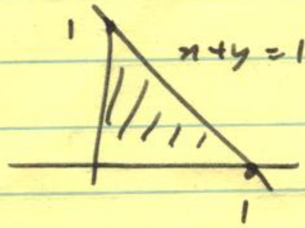
20E Spring 2012 MT 1 solutions

① 

$$\int_0^1 dx \int_0^{1-x} dy \,.\, x e^{-y}$$

$$= \int_0^1 dx. \left[-x e^{-y} \right]_{y=0}^{y=1-x}$$

$$= \int_0^1 dx \left(-x e^{x-1} + x \right)$$

$$= \frac{-1}{e} \cdot \left(\int_0^1 x e^{x} dx \right) + \frac{1}{2}$$

$$\int_0^1 x e^x dx = \left[x e^x \right]_0^1 - \int_0^1 1.e^x dx = e - (e-1) = 1$$

so integral is $\underline{\underline{\frac{-1}{e} + \frac{1}{2}}}$

② (This is a question from section 7.1, which we haven't covered yet & can ignore!
But here is the solution anyway :

$$\underline{c}(t) = \begin{pmatrix} t \\ \frac{2}{3} t^{3/2} \\ t \end{pmatrix} \quad \text{so} \quad \frac{d\underline{c}}{dt} = \begin{pmatrix} 1 \\ t^{1/2} \\ 1 \end{pmatrix} \quad \left\| \frac{d\underline{c}}{dt} \right\| = \sqrt{2+t}$$

$$\text{so} \int_\gamma z \, ds = \int_2^7 dt \,.\, t \cdot \sqrt{2+t} = \int_0^5 du \, (u-2)\sqrt{u}$$

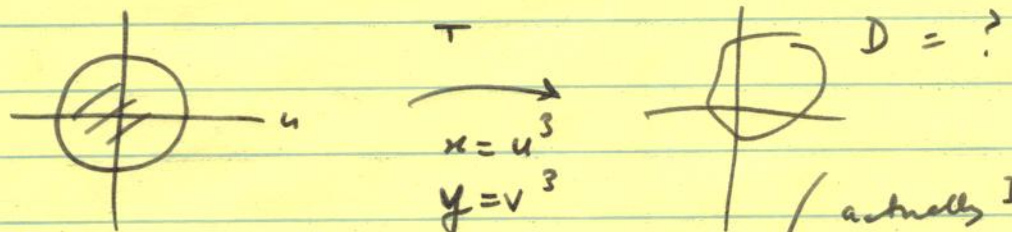
$u = t+2$

$$= \left[\frac{2u^{5/2}}{5} - \frac{2.2\,u^{3/2}}{3} \right]_0^5$$

$$= \frac{2}{5} \cdot 25\sqrt{5} - \frac{4}{3} \cdot 5\sqrt{5} \quad = \quad \frac{10\sqrt{5}}{3}$$

✓

3



$D^* =$ unit circle

(actually D is the curve $x^{2/3} + y^{2/3} = 1$ which looks like this:

but we don't need to know that!)

$$\text{Area}(D) = \int_D 1 \; dx\, dy$$

$$= \int_{D^*} 1 . \; |\det DT| \; du\, dv$$

Now $DT = \begin{pmatrix} 3u^2 & 0 \\ 0 & 3v^2 \end{pmatrix}$

$$\text{so area} = \int_{D^*} 9u^2v^2 \; du\, dv$$

Jacobian for polar coord change.

$$= (\text{change to polar}) \int_0^1 dr \int_0^{2\pi} d\theta \, . \; 9r^4 \cos^2\theta \sin^2\theta \, . \; r$$

$$= \left[\frac{9r^6}{6} \right]_0^1 . \int_0^{2\pi} \cos^2\theta \sin^2\theta \; d\theta$$

$$= \frac{3}{2} . \int_0^{2\pi} \frac{1}{4} \sin^2 2\theta \;\; d\theta \qquad (\sin 2\theta = 2\cos\theta \sin\theta)$$

$$= \frac{3}{2} . \int_0^{2\pi} \frac{1}{8} (1 - \cos 4\theta) \, d\theta \qquad \left(\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta) \right)$$

$$= \frac{3}{2} \cdot \frac{1}{8} \cdot \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{2\pi}$$

$$= \frac{3}{8}\pi$$

2013 MT1 SOLUTIONS

(1)

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^1$$

$$(0,0,0) \longmapsto (2,3)$$

$$f(x,y) = xy - x + y$$

$$Dg\Big|_{(0,0,0)} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Calculate $Df = \begin{pmatrix} y-1 & x+1 \end{pmatrix}$

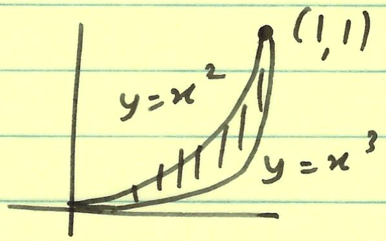
$$\therefore Df\Big|_{(2,3)} = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

$\therefore$ Chain rule $D(f \circ g)\Big|_{(0,0,0)} = Df\Big|_{(2,3)} \cdot Dg\Big|_{(0,0,0)}$

$$= \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 14 & 19 & 24 \end{pmatrix}}.$$

(2)



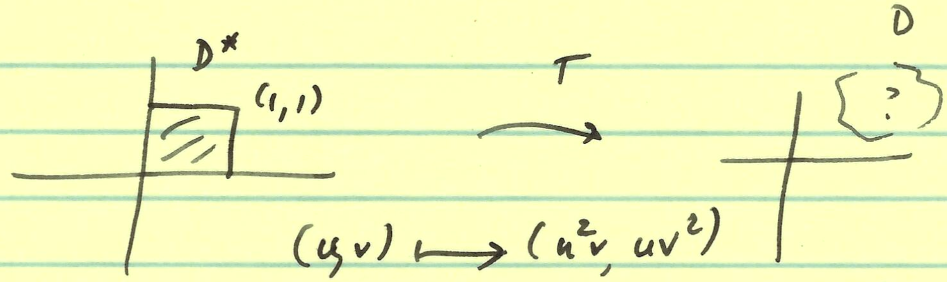
Slicing gives $\displaystyle\int_0^1 dx \int_{x^3}^{x^2} dy \; xy$

$$= \int_0^1 dx \left[\frac{xy^2}{2}\right]_{x^3}^{x^2}$$

$$= \int_0^1 dx \left(\frac{x^5}{2} - \frac{x^7}{2}\right)$$

$$= \left[\frac{x^6}{12} - \frac{x^8}{16}\right]_0^1 = \frac{1}{12} - \frac{1}{16} = \underline{\underline{\frac{1}{48}}}$$

3



$$(u,v) \longmapsto (u^2v,\ uv^2)$$

$$DT = \begin{array}{c c} & \begin{array}{cc} u & v \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 2uv & u^2 \\ v^2 & 2uv \end{pmatrix} \end{array}$$

$$\therefore |\mathrm{Jac}| = 3u^2v^2$$

So $$\iint_D 1 \, dx\, dy = \iint_{D^*} 1.\ 3u^2v^2 \, du\, dv$$

$$= \int_0^1 du \int_0^1 dv\ 3u^2v^2$$

$$= 3 \cdot \frac{1}{3} \cdot \frac{1}{3} = \underline{\underline{\frac{1}{3}}}$$

4 $f(x,y) = x \log y$ [NOTE: I write "log" for natural logarithm. If I mean any other base I'll always write "$\log_{10}$" etc...]

$$Df = \begin{pmatrix} \log y & \frac{x}{y} \end{pmatrix}$$

$$D^2f = \begin{pmatrix} 0 & 1/y \\ 1/y & -x/y^2 \end{pmatrix}$$

so at $(1,1)$, $f(1,1) = 0$ $\quad Df\big|_{(1,1)} = (0 \quad 1)$

and $$D^2f\big|_{(1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

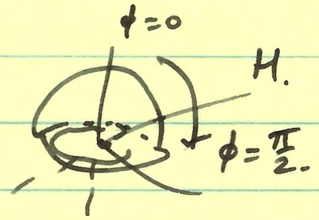
$$\therefore f(1+h_1, 1+h_2) \approx 0 + (0.h_1 + 1.h_2) + \frac{1}{2}\left(2h_1h_2 - h_2^2\right)$$

$$\therefore f(1.03, 1.02) \approx 0.02 + \frac{1}{2}\left(2.(0.03)(0.02) - (0.0004)\right)$$

$$= \underline{0.0204}$$

2014 MT 1 SOLUTIONS

1. (Remember: I said I won't put any triple integrals on the MT!)



In spherical polar coords H is the region
$0 \le \rho \le 1$
$0 \le \phi \le \pi/2$
$0 \le \theta \le 2\pi$.

(NB: we don't use negative values of ϕ if we allow θ to go all the way round, otherwise we'd be duplicating coordinates of points)

Also we have $z=\rho\cos\phi$ and
$|Jac| = \rho^2 \sin\phi$

so $$\iiint_H z^2\, dV = \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi\, (\rho\cos\phi)^2 \cdot \rho^2 \sin\phi$$

$$= 2\pi \times \int_0^1 \rho^4\, d\rho \times \int_0^{\pi/2} \cos^2\phi \sin\phi\, d\phi$$

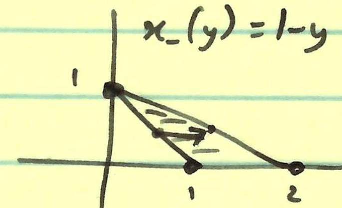
$$= 2\pi \cdot \frac{1}{5} \cdot \left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}$$

$$= 2\pi/15$$

(2)

$x_+(y) = 2-2y$

$x_-(y) = 1-y$



D

(2) Use horizontal slices to make the limits simpler. (all one range rather than two parts).

$$\int_0^1 dy \int_{1-y}^{2-2y} dx \, . \, xy = \int_0^1 dy \left[\frac{x^2}{2} y \right]_{x=1-y}^{x=2-2y}$$

$$= \int_0^1 dy \left[3 \; \frac{(1-y)^2}{2} y \right]$$

$$= \int_0^1 dy \; \frac{3}{2} \left(y - 2y^2 + y^3 \right)$$

$$= \frac{3}{2} \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1$$

$$= \frac{3}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{8}$$

(3) $f(x,y) = \dfrac{1}{x^2+y^2}$ $\therefore$ f at $(1,1) = \dfrac{1}{2}$.

$$Df = \left(\frac{-1}{(x^2+y^2)^2} \cdot 2x \qquad \frac{-1}{(x^2+y^2)^2} \cdot 2y \right)$$

Df at $(1,1) = \left(-\frac{1}{2} \quad -\frac{1}{2} \right)$

$$D^2 f = \begin{pmatrix} \frac{2}{(x^2+y^2)^3} \cdot 2x \cdot 2x - \frac{2}{(x^2+y^2)^2} & \frac{2}{(x^2+y^2)^3} \cdot 2x \cdot 2y \\ \frac{2}{(x^2+y^2)^3} \cdot 2y \cdot 2x & \frac{2}{(x^2+y^2)^3} \cdot 2y \cdot 2y - \frac{2}{(x^2+y^2)^2} \end{pmatrix}$$

I'm going to write $r^2 = x^2+y^2$ as an abbreviation; therefore

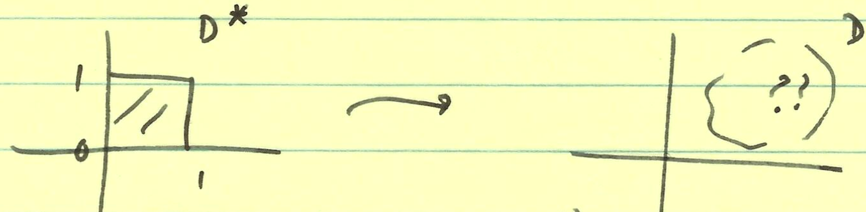
$$D^2 f = \frac{2}{r^6}\begin{pmatrix} 4x^2-r^2 & 4xy \\ 4xy & 4y^2-r^2 \end{pmatrix}$$

$\therefore$ at $(1,1)$ (where $r^2=2$)
$$= \frac{2}{2^3}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$$

$\therefore$ Taylor expansion is

$$f(1+h_1, 1+h_2) = \frac{1}{2} + \left(-\frac{1}{2}h_1, -\frac{1}{2}h_2\right) + \frac{1}{2}\left(\frac{1}{2}h_1^2 + \frac{1}{2}h_2^2 + 2h_1h_2\right)$$

(4)



$$T(u,v) = \left(e^{u+v}, e^{u-v}\right)$$

Area of D is $\iint_D 1 \, dx\, dy$

$$DT = \begin{array}{c c} & \begin{array}{cc} u & v \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} e^{u+v} & e^{u+v} \\ e^{u-v} & -e^{u-v} \end{pmatrix} \end{array}$$

so Jacobian is
$$e^{u+v}(-e^{u-v}) - e^{u-v}.e^{u+v}$$
$$= -e^{2u} - e^{2u} = \underline{-2e^{2u}}$$
and $|Jac| = +2e^{2u}$

$\therefore$ changing variables, $= \iint_{D^*} 1.\ 2e^{2u}\, du\, dv.$

$$= \int_0^1 dv \int_0^1 du\ .\ 2e^{2u} = 1 \times \left[e^{2u}\right]_0^1$$
$$= \underline{\underline{e^2 - 1}}$$

20E MT1 2016 SOLUTIONS (not guaranteed!)

1. $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$

$(1,2) \longmapsto f(1,2) \longmapsto f(f(1,2))$

$f(1,2) = (2,5)$, $f(f(1,2)) = f(2,5) = (10, 27)$.

$f(x,y) = (xy,\ x+y^2)$

$$Df = \begin{pmatrix} y & x \\ 1 & 2y \end{pmatrix}$$

$$Dg\Big|_{(1,2)} = Df\Big|_{(2,5)} \cdot Df\Big|_{(1,2)}$$

$$= \begin{pmatrix} 5 & 2 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 12 & 13 \\ 12 & 41 \end{pmatrix}}}$$

2. $f(x,y) = \sin(x^2 y)$ $\quad (= 1$ at $(1, \pi/2))$

$$\therefore Df(x,y) = \left(\cos(x^2y)\cdot 2xy,\quad \cos(x^2y)\cdot x^2 \right)$$

$$(= (0,0) \text{ at } (1, \pi/2))$$

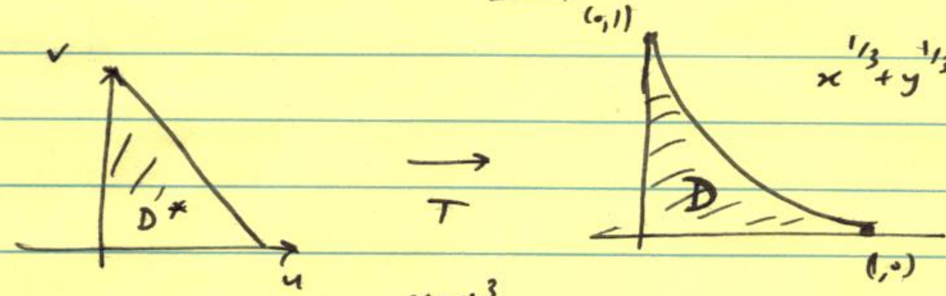
$$D^2 f(x,y) = \begin{pmatrix} -\sin(x^2y)\cdot(2xy)^2 + \cos(x^2y)\cdot 2y & -\sin(x^2y)\cdot 2xy\cdot x^2 + \cos(x^2y)\cdot 2x \\ -\sin(x^2y)\cdot 2xy\cdot x^2 + \cos(x^2y)\cdot 2x & -\sin(x^2y)\cdot x^2\cdot x^2 + 0 \end{pmatrix}$$

At $(1, \pi/2)$ $\sin(x^2y) = 1$, $\cos(x^2y) = 0$ so get

$$D^2 f\Big|_{(1,\frac{\pi}{2})} = \begin{pmatrix} -\pi^2 & -\pi \\ -\pi & -1 \end{pmatrix}$$

so $f\left(1+h_1, \frac{\pi}{2}+h_2\right) \approx 1 + \frac{1}{2}\left(-\pi^2 h_1^2 - 2\pi h_1 h_2 - h_2^2\right)$

3



$x = u^3$

$y = v^3$

(We don't really need a clear picture of the region D, but we can observe that the boundaries

$x \geq 0,\ y \geq 0,\ x^{1/3}+y^{1/3}=1$ become $u \geq 0,\ v \geq 0,\ u+v=1$

under the substitution, so the triangle D^* is clear)

(Note also that T is a bijective function $D^* \leftrightarrow D$ because the substitution is clearly invertible, using $u = x^{1/3},\ v = y^{1/3}$)

const fn of x & y ~~becomes~~ becomes const fn of u & v

$$\text{Area}(D) = \iint_D 1 \, dx\, dy = \iint_{D^*} 1 \; \left|\text{Jac } T\right| \, du\, dv$$

Now $DT = \begin{pmatrix} 3u^2 & 0 \\ 0 & 3v^2 \end{pmatrix}$ so this factor is just $9u^2v^2$

$$\therefore \text{ area} = \iint_{D^*} 9u^2v^2 \, du\, dv = \int_0^1 du \cdot 9u^2 \int_0^{1-u} dv \cdot v^2$$

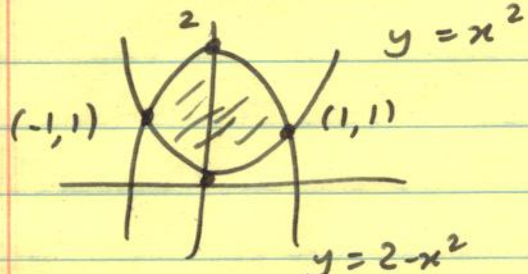
(do by slicing)

$$= \int_0^1 du \cdot 9u^2 \cdot \left[\frac{v^3}{3}\right]_0^{1-u}$$

$$= \int_0^1 du \; 9u^2 \frac{(1-u)^3}{3} = \int_0^1 du \left(3u^2 - 9u^3 + 9u^4 - 3u^5\right)$$

$$= \frac{3}{3} - \frac{9}{4} + \frac{9}{5} - \frac{3}{6} = \frac{1}{20}$$

20E Winter 2017 Midterm 1 solutions

① 

$$\int_{-1}^{1} dx \int_{x^2}^{2-x^2} dy \cdot x^2 y$$

$$= \int_{-1}^{1} dx \cdot \left[\frac{x^2 y^2}{2} \right]_{y=x^2}^{y=2-x^2}$$

$$= \int_{-1}^{1} dx \left(\frac{x^2(4-4x^2+x^4)}{2} - \frac{x^6}{2} \right)$$

$$= \int_{-1}^{1} dx \, (2x^2 - 2x^4) = \left[\frac{2x^3}{3} - \frac{2x^5}{5} \right]_{-1}^{1}$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right) - \left(\frac{-2}{3} - \left(\frac{-2}{5}\right)\right) = \frac{4}{15} - \left(\frac{-4}{15}\right) = \underline{\underline{\frac{8}{15}}}.$$

②

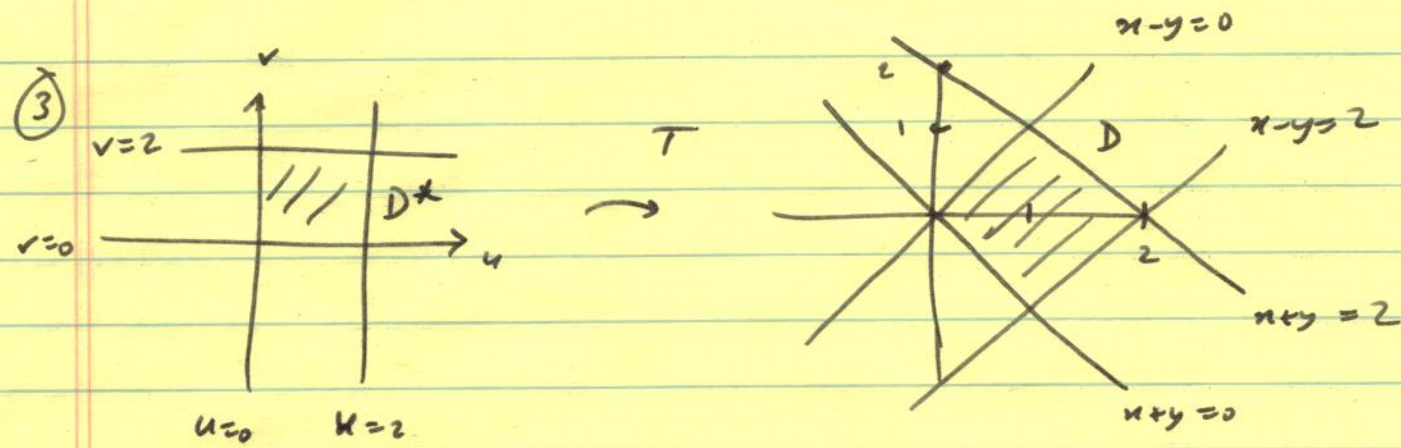
$$f(x,y) = e^{3x-2y} \quad \text{at } (1,1) \quad = e$$

$$Df(x,y) = \begin{pmatrix} 3e^{3x-2y} & -2e^{3x-2y} \end{pmatrix} \text{ at } (1,1) = \begin{pmatrix} 3e & -2e \end{pmatrix}$$

$$D^2 f(x,y) = \begin{pmatrix} 9e^{3x-2y} & -6 \cdot e^{3x-2y} \\ -6e^{3x-2y} & (-2)^2 e^{3x-2y} \end{pmatrix} \text{ at } (1,1) = \begin{pmatrix} 9e & -6e \\ -6e & 4e \end{pmatrix}$$

so $f(1+h_1, 1+h_2) \approx e + (3e \cdot h_1 - 2e \cdot h_2) + \frac{1}{2}\left(9e\,h_1^2 - 12e\,h_1 h_2 + 4e\,h_2^2\right)$

(3)



The transformation changes the four lines

$x+y=0$ (etc) into $u=0$ (etc)

which are easy to draw in the u-v plane

(we don't really <u>need</u> to draw D in the xy plane)

Writing $\begin{cases} u = x+y \\ v = x-y \end{cases}$ is giving the formula for the

<u>inverse</u> transformation T^{-1}, we need to

rewrite it as $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$ to get the formula for T itself

Then $DT = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

so the factor $|\det DT|$ is $\left|-\frac{1}{2}\right| = +\frac{1}{2}$

$$\therefore \int_D (x+y)\, e^{x^2-y^2}\, dx\, dy = \int_{D^*} u\, e^{uv} \cdot \frac{1}{2} \cdot du\, dv$$

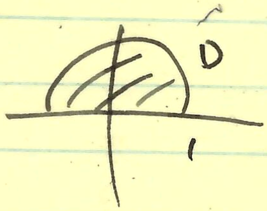
$$= \frac{1}{2}\int_0^2 du \int_0^2 dv \cdot u e^{uv} = \frac{1}{2}\int_0^2 du \cdot \Big[e^{uv}\Big]_{v=0}^{v=2}$$

$$= \frac{1}{2} \int_0^2 du \cdot \left(e^{2u} - 1 \right)$$

$$= \frac{1}{2} \cdot \left[\frac{e^{2u}}{2} - u \right]_0^2$$

$$= \frac{1}{2} \left(\frac{e^4}{2} - 2 - \frac{1}{2} \right) = \frac{e^4 - 5}{4}$$

20E Fall 2017 Midterm 1

① 

$$\int_D y \, dx\, dy = \int_0^1 dr \int_0^\pi d\theta \cdot \underbrace{r\sin\theta}_{`y'} \cdot \underbrace{r}_{\text{Jacobian}}$$

$$= \left[\frac{r^3}{3}\right]_0^1 \cdot \left[-\cos\theta\right]_0^\pi$$

$$= \frac{1}{3} \cdot 2$$

The average is obtained by dividing by the area of D, which is simply $\frac{1}{2}\cdot\pi$ (by geometry)

$\therefore$ average is $\dfrac{4}{3\pi}$

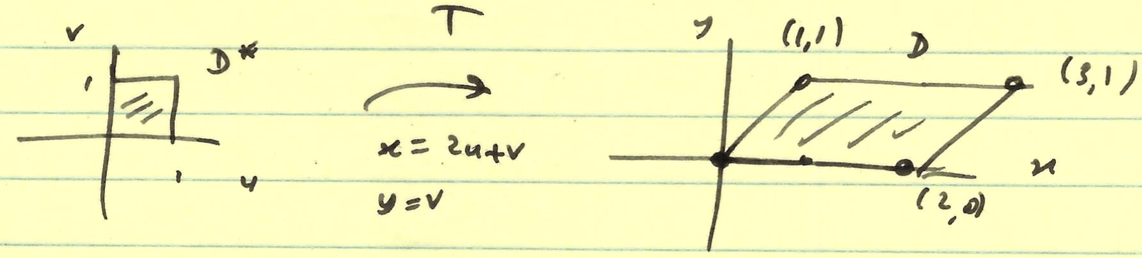
② $f(x,y) = \sin(xy)$ at $(1, \pi/2)$ is 1

$$Df = \begin{pmatrix} y\cos(xy) & x\cos(xy) \end{pmatrix} \text{ at } (1,\pi/2) \text{ is } \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$D^2 f = \begin{pmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{pmatrix} \text{ at } (1,\pi/2) \text{ is } \begin{pmatrix} -\frac{\pi^2}{4} & -\frac{\pi}{2} \\ -\frac{\pi}{2} & -1 \end{pmatrix}$$

So $f(1+h_1, \frac{\pi}{2}+h_2) = 1 + \frac{1}{2}\left(-\frac{\pi^2}{4}\cdot h_1^2 - \pi\cdot h_1 h_2 - h_2^2\right)$

3



The transformation is linear and takes

$(1,0) \mapsto (2,0)$ and $(0,1) \mapsto (1,1)$, hence

D^* is simply the unit square in the u-v plane.

The Jacobian is $\begin{array}{c|cc|} & \partial/\partial u & \partial/\partial v \\ x & 2 & 1 \\ y & 0 & 1 \end{array} = 2$

The integrand is $(x-y)y = 2u \cdot v$

$$\therefore \int_D xy - y^2 \, dx\,dy = \int_0^1 du \int_0^1 dv \;\; 2uv \cdot 2$$

$$= 4 \left[\frac{u^2}{2}\right]_0^1 \left[\frac{v^2}{2}\right]_0^1 = \underline{\underline{1}}$$