Fall 2016

1) $f(x,y) = e^{x^2+y}$.

a) The tangent plane at the point $(0,0,1)$ is given by

$$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0).$$
$$= 1 + f_x(0,0)\,x + f_y(0,0)\,y$$

We need to find $f_x(0,0)$ and $f_y(0,0)$.

$f_x(x,y) = 2xe^{x^2+y}$ and $f_y(x,y) = e^{x^2+y}$

$f_x(0,0) = 0$ $\qquad$ $f_y(0,0) = 1$.

$\therefore$ The tangent plane at $(0,0,1)$ is

$$z = 1 + y.$$

(*) b) There will be no Taylor's expansion on our exam.

2) Let $f(u,v) = (\cos u, v + \sin u)$

and $g(x,y,z) = (x^2+\pi y^2, xz)$.

a)
$$Dg(x,y,z) = \begin{bmatrix} \frac{\partial(x^2+\pi y^2)}{\partial x} & \frac{\partial(x^2+\pi y^2)}{\partial y} & \frac{\partial(x^2+\pi y^2)}{\partial z} \\ \frac{\partial(xz)}{\partial x} & \frac{\partial(xz)}{\partial y} & \frac{\partial(xz)}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 2\pi y & 0 \\ z & 0 & x \end{bmatrix}$$

b). By the chain rule,

$$D(f\circ g)(0,1,1) = Df(g(0,1,1))\, Dg(0,1,1)$$

$$= Df(\pi,0)\, Dg(0,1,1)$$

$$= \begin{bmatrix} \dfrac{\partial(\cos u)}{\partial u} & \dfrac{\partial(\cos u)}{\partial v} \\ \dfrac{\partial(v+\sin u)}{\partial u} & \dfrac{\partial(v+\sin u)}{\partial v} \end{bmatrix}_{(u,v)=(\pi,0)} \cdot \begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

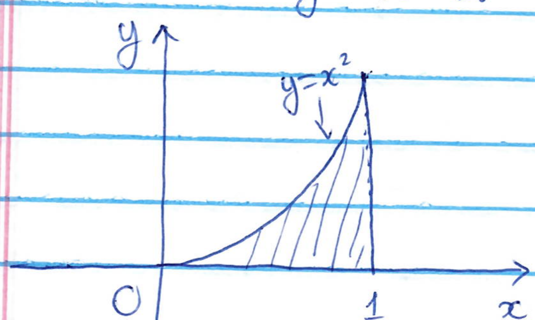
$$= \begin{bmatrix} -\sin u & 0 \\ \cos u & 1 \end{bmatrix}_{(u,v)=(\pi,0)} \begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\pi & 0 \\ \cos\pi & 1 \end{bmatrix} \begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2\pi & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2\pi & 0 \end{bmatrix}.$$

3) $D = \left\{ \begin{matrix} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{matrix} \right\}.$



$$\int_0^1 \int_{\sqrt{y}}^1 (x^2 + y^3)\, dx\, dy$$

$$= \int_0^1 \int_0^{x^2} (x^2 + y^3)\, dy\, dx$$

$$= \int_0^1 x^2 y + \frac{y^4}{4} \Big|_{y=0}^{x^2} dx$$

$$= \int_0^1 x^4 + \frac{x^8}{4}\, dx = \frac{x^5}{5} + \frac{x^9}{36}\Big|_{x=0}^{1} = \frac{1}{5} + \frac{1}{36} = \frac{41}{180}.$$

4) Let $D^* = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(u,v) = (u^2 v, uv^2)$.

Recall that

$$\text{Area}(D) = \iint_D dx\,dy.$$

Since $D = T(D^*)$, we can use change of variables.

$$\text{Area}(D) = \iint_D dx\,dy = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

$$= \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| \cancel{du}\,dv$$

$$= \int_0^1\int_0^1 \left| \begin{matrix} 2uv & u^2 \\ v^2 & 2uv \end{matrix} \right| du\,dv$$

$$= \int_0^1\int_0^1 |4u^2v^2 - u^2v^2|\, du\,dv$$

$$= \int_0^1\int_0^1 3u^2v^2\, du\,dv$$

$$= \frac{1}{3}.$$