8.4 Gauss' Theorem

- Divergence of a vector field.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\mathrm{div}(\vec{F}) = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(F_1, F_2, F_3)$$

$$\boxed{\mathrm{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

(dot product of ∇ with $\vec{F}$).

E.g. Compute the divergence of

$$\vec{F} = x^2\vec{i} + xyz\vec{j} + y\vec{k}.$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(y).$$

$$= 2x + xz + 0.$$

$$= 2x + xz.$$

Physical interpretation: If $\vec{F}$ is the velocity field of a fluid, then $\nabla\cdot\vec{F}$ is the rate of expansion per unit volume under the ~~fluid of~~ flow of the fluid.

Thm: $\nabla\cdot(\nabla\times\vec{F}) = 0$ (Check it using the definition).

* Gauss' Thm: If S is a closed surface bounding a region W, with normal pointing outward, and if $\vec{F}$ is a vector field defined over W and differentiable:

$$\iint_{\substack{S \\ (\partial W)}} \overbrace{\vec{F}\cdot d\vec{S}}^{\vec{F}\cdot\vec{n}\,dS} = \iiint_W \nabla\cdot\vec{F}\,\underbrace{dV}_{dxdydz}$$

E.g: $\vec{F} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$.

S: unit sphere.

Evaluate $\iint_S \vec{F}\cdot d\vec{S}$.

Sol: $$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \nabla\cdot\vec{F}\,dV = \iiint_{Ball} (2 + 2y + 2z)\,dxdydz$$

$$= 2\iiint_{Ball} dV + \cancel{2\iiint_{Ball} y\,dV}^{\,0} + \cancel{2\iiint_{Ball} z\,dV}^{\,0}$$

$$= 2\cdot\frac{4\pi}{3}\cdot 1^3$$

$$= \frac{8\pi}{3}$$

* Interpretation: At a point (x,y,z), $(\nabla\cdot\vec{F})\big|_{(x,y,z)}$ is the rate of outward flow (flux) per unit volume (or expansion). So, in a sense, Gauss' theorem tells us that the total outward flow $\iiint_W (\nabla\cdot\vec{F})\,dV$ equals the total flux out of the boundary.

E.g. Compute $\iint_S \vec{F}\cdot d\vec{S}$ where S is the surface of the box: $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ and $\vec{F} = (3x + e^{-yz})\vec{i} + (y^2 + (5\sinh(z+x)))\vec{j} + (xz + y^{72})\vec{k}$.

Sol: by Gauss' Thm $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\,dV$.

$$= \iiint_W (3 + 2y + x)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (3 + 2y + x)\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^1 \left(3 + 2y + \frac{1}{2}\right) dy\,dz$$

$$= \int_0^1 (3.5 + 1) \cancel{+\tfrac{1}{2}}\ dz$$

$$= 4.5.$$

Remark: Field is terrible but $\nabla\cdot\vec{F}$ is nice.
$\Rightarrow$ Gauss' thm's helpful.

- Applications (Physics)

Given a charge density $\rho(x,y,z)$ in a region W, the field $\vec{E}$ satisfies $\nabla\cdot\vec{E} = \rho$ (given)

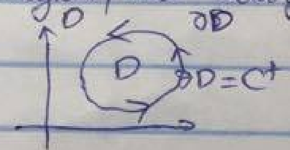
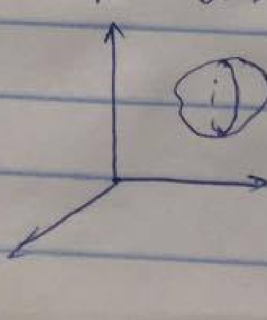
$$\Rightarrow \underbrace{\iiint_W \nabla\cdot\vec{E}\,dV}_{\text{total charge } Q \text{ inside } W} = \underbrace{\iint_S \vec{E}\cdot d\vec{S}}_{\text{flux out of the surface.}}$$

Remark: A two dimensional version of Gauss divergence thm, i.e., with $\vec{F} = P\vec{i} + Q\vec{j}$ and region D bounded by closed curve $C = \partial D$ is

$$\boxed{\iint_D (\nabla \cdot \vec{F})\, dx\, dy = \int_C (\vec{F} \cdot \vec{n})\, ds.}$$

This is the divergence form of Green's thm.

- Review of main thms:

Thm	Applies in	Applies to	States that
FTOLI	2D & 3D	Curves/boundary of curve 	$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$
Green's thm	2D	Regions D / their bdary ∂D 	$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx\, dy = \int_{\partial D} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial D} P\, dx + Q\, dy$ $\Leftrightarrow \iint_D (\nabla \cdot \vec{F})\, dx\, dy = \int_{\partial D} \vec{F} \cdot \vec{n}\, ds$
Stoke's thm	3D	Surfaces S / their bdary ∂S 	$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_{\vec{c}} \vec{F} \cdot \vec{c}\,'(t)\, dt$
Gauss' thm	3D	Regions W / their bdary $\partial W = S$ 	$\iiint_W (\nabla \cdot \vec{F})\, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$

Section 8.3 Consertive Vector Fields.

FTLI: $\int_{\vec{c}} (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.

If the field $\vec{F}$ is a gradient vector field, ie. $\vec{F} = \nabla f$ for some function $f(x,y,z)$, then the line integral is path independent.

E.g. $\vec{F} = \left(\cos(x)\cos(y) + yze^{xyz},\ -\sin(x)\sin(y) + xze^{xyz},\ xye^{xyz}\right)$

$\vec{c}(t) = \left(\cos(2\pi t), \sin(2\pi t), t^2, t\right), \quad 0 \le t \le 1$.

Evaluate $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

Sol: $\vec{F} = \nabla f$ where $f = \sin(x)\cos(y) + \cancel{xyze}\ e^{xyz}$.

$$\Rightarrow \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(1)) - f(\vec{c}(0))$$
$$= f(\cos(2\pi) \sin(2\pi), 1, 1) - f(0,0,0)$$
$$= f(0,1,1) - f(0,0,0)$$
$$= \sin(0)\cos(1) + e^0 - 1$$
$$= 0.$$

(much easier than $\int_0^1 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$).

$\Rightarrow$ would like to know when vector fields are gradients

Thm: Let $\vec{F}$ have continuous partial derivatives.
All these statements are equivalent:

(i) $\int_{C} \vec{F} \cdot d\vec{s} = 0$ for all oriented simple closed curves.

(ii) $\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$ for simple oriented curves with the same endpoints.

(iii) $\vec{F} = \nabla f$ for some f.

(iv) $\nabla \times \vec{F} = 0$

$\Rightarrow$ we call such a field $\vec{F}$ <u>conservative</u>.

<u>In</u> the previous example, how did we know that $\vec{F}$ was conservative? How did we find f?

<u>Answer:</u> 1) by inspection.

or 2) $$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x)\cos(y) + yze^{xyz} & -\sin(x)\sin(y) + xze^{xyz} & xye^{xyz} \end{vmatrix}$$

$$= \left(xe^{xyz} + x^2yze^{xyz} - \left(xe^{xyz} + x^2yze^{xyz}\right)\right)\vec{i}$$
$$- \left(ye^{xyz} + xy^2ze^{xyz} - \left(ye^{xyz} + xy^2ze^{xyz}\right)\right)\vec{j}$$
$$+ \left(\left(-\cos(x)\sin(y) + ze^{xyz} + xyz^2e^{xyz}\right) - \left(-\cos(x)\sin(y) + ze^{xyz} + xyz^2e^{xyz}\right)\right.$$
$$= 0.$$

3) If such f exists; then

$$\frac{\partial f}{\partial x} = \cos x \cos y + yze^{xyz}$$

$$\frac{\partial f}{\partial y} = -\sin(x)\sin y + xze^{xyz}$$

$$\frac{\partial f}{\partial z} = xye^{xyz} \Rightarrow f(x,y,z) = \int xye^{xyz}\,dz + g(x,y)$$
$$= e^{xyz} + g(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = xze^{xyz} + \frac{\partial g}{\partial y}.$$

$$\Rightarrow \frac{\partial g}{\partial y} = -\sin x \sin y.$$

$$\Rightarrow g(x,y) = -\int \sin x \sin y\,dy + h(x)$$
$$= \sin x \cos y + h(x).$$

$$\Rightarrow f(x,y,z) = e^{xyz} + \sin x \cos y + h(x).$$

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$\Rightarrow \quad \frac{\partial f}{\partial x} = yz e^{xyz} + \cos x \cos y + h'(x).$

$\Rightarrow \quad h'(x) = 0 \quad \Rightarrow \quad h(x) = 0$ works.

~~E.g. $\vec{F} = (2xy - \sin x)\vec{i} + x^2$~~

<u>Remark.</u> In 2 dimensions (i.e. the plane)

$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

$\Rightarrow \vec{F} = P\vec{i} + Q\vec{j}$ is a conservative when $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

($\Rightarrow$ there is an f: $\nabla f = \vec{F}$).

<u>E.g.</u> $\vec{F} = (2xy - \sin x)\vec{i} + x^2\vec{j}$.

is a conservative because $\frac{\partial (x^2)}{\partial x} = 2x$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$

and $\frac{\partial (2xy - \sin x)}{\partial y} = 2x$

How to find f s.t. $\nabla f = \vec{F}$?

$\frac{\partial f}{\partial x} = 2xy - \sin x \qquad \frac{\partial f}{\partial y} = x^2.$

$\Rightarrow \quad f(x,y) = x^2 y + \cos x + g(y).$

$\frac{\partial f}{\partial y} = x^2 + g'(y). \quad \rightarrow \quad g'(y) = 0 \Rightarrow$ take $g(y) = 0.$

$\Rightarrow \quad f(x,y) = x^2 y + \cos x.$

<u>Thm:</u> Let $\vec{F}$ have continuous partial derivatives.
All the following statements are equivalent:

i) $\int_C \vec{F}\cdot d\vec{s} = 0$ for all oriented simple <u>closed</u> curve.

ii) $\int_{\vec{C}_1} \vec{F}\cdot d\vec{s} = \int_{\vec{C}_2} \vec{F}\cdot d\vec{s}$ for simple oriented curves with the same endpoints.

iii) $\vec{F} = \nabla f$ for some f.

iv) $\nabla \times \vec{F} = 0$.

We call such a field $\vec{F}$ <u>conservative</u>.

* <u>Remark:</u> In 2 dimensions (i.e. planar case).

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

So $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative when

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad \left(\text{so there is an } f\text{: } \nabla f = \vec{F}\right).$$

<u>E.g.</u> $\vec{F} = (2xy - \sin x)\,\vec{i} + x^2\vec{j}$.

is conservative when $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$$2x = 2x$$

To find f, we solve

$$\frac{\partial f}{\partial x} = 2xy - \sin x \qquad \frac{\partial f}{\partial y} = x^2$$

$$\Rightarrow \quad f(x,y) = x^2 y + \cos x$$

Recall: If $\nabla\times\vec{F} = 0$, then there exists f such that $\vec{F} = \nabla f$.
It's also true that if $\nabla\cdot\vec{F} = 0$, there exists $\vec{G}$ such that $\vec{F} = \nabla\times\vec{G}$.

* Maxwell's Equations: govern the propagation of electromagnetic fields.

. Relate: $\vec{E}(x,y,z,t)$ the electric fields.
$\vec{H}(x,y,z,t)$ the magnetic fields.
to each other and to
$\rho(x,y,z,t)$ the charge density.
and $\vec{J}(x,y,z,t)$ the current density.

(In the simplest form) they are

(1) $\nabla\cdot\vec{E} = \rho$ Gauss' law.

(2) $\nabla\cdot\vec{H} = 0$

(3) $\nabla\times\vec{E} + \frac{\partial\vec{H}}{\partial t} = 0$ Faraday's law.

(4) $\nabla\times\vec{H} - \frac{\partial\vec{E}}{\partial t} = \vec{J}$ Ampere's law.

E.g. Let S be a surface with boundary C. Then

$$\int_C \vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\iint_S \vec{H}\cdot d\vec{S}$$

(Remark: $\int_C \vec{E}\cdot d\vec{s}$ = voltage around C

and $\iint_S \vec{H}\cdot d\vec{S}$ = magnetic flux across S).

Sol: By Stoke's thm,

$$\int_C \vec{E}\cdot d\vec{s} = \iint_S \nabla\times\vec{E}\cdot d\vec{S} \overset{\text{Maxwells eq. (3)}}{=} \iint_S -\frac{\partial\vec{H}}{\partial t}\cdot d\vec{S}.$$

Here, $\nabla^2\vec{E} = \nabla^2(E_x)\vec{i} + \nabla^2(E_y)\vec{j} + \nabla^2(E_z)\vec{k}$ (vector laplacian)

E_x, E_y, and E_z are the components of $\vec{E}$.

(See Section 4.4).

$$\Rightarrow \nabla(\nabla\cdot\vec{E})^{\nearrow 0} - \nabla^2\vec{E} = -\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2\vec{E}}{\partial t^2} = \nabla^2\vec{E}.$$

Similarly, $\frac{\partial^2 H}{\partial t^2} = \nabla^2\vec{H}.$

E.g. The gravitational force field: of a mass m located at $\vec{r}(x,y,z) = (x,y,z)$ is

$$\vec{F}(x,y,z) = -\frac{GmM}{r^3}\vec{r}$$

M and G are constants and $r = \|\vec{r}\|$.

$\cancel{\nabla\times F = \nabla}$

We need the following identity (See section 4.4)

$$\nabla\times(\underbrace{g}_{\text{scalar function}}\vec{G}) = g(\nabla\times\vec{G}) + (\nabla g)\times\vec{G}.$$

Then

$$\nabla\times\vec{F} = \nabla\times\left(-\frac{GmM}{r^3}\vec{r}\right) = -\frac{GmM}{r^3}\left[\nabla\times\vec{r} + \nabla\left(\frac{\overset{1}{\cancel{GmM}}}{r^3}\right)\times\cancel{G}\,\vec{r}\right]$$

we can show that $\nabla\left(\frac{1}{r^3}\right) = 0$ and $\nabla\times\vec{r} = 0.$

$\Rightarrow \nabla\times\vec{F} = 0.$

$\Rightarrow \vec{F}$ is conservative.

$$= -\frac{\partial}{\partial t}\iint_S \vec{H}\cdot d\vec{S}$$ since the integral is with respect to space (x,y,z), we can bring $\frac{\partial}{\partial t}$ out.

E.g: Let S be a surface with boundary ∂S. Suppose $\vec{E}$ is an electric field that is perpendicular to ∂S. Show that magnetic flux across S is constant in time.

Sol: Let ∂S be parametrized by $c(\theta) = (x(\theta), y(\theta), z(\theta))$. $a \le \theta \le b$.

Recall magnetic flux $= \iint_S \vec{H}\cdot d\vec{S}$.

By the previous example,

$$\frac{\partial}{\partial t}\iint_S \vec{H}\cdot d\vec{S} = -\int_{\partial S} \vec{E}\cdot d\vec{s} = -\int_a^b \vec{E}(\vec{c}(\theta))\cdot \vec{c}'(\theta)\, d\theta.$$

$$= 0$$

since $\vec{E}(c(\theta))\cdot \vec{c}'(\theta) = 0$, i.e., $\vec{E}$ is perpendicular to ∂S

E.g. Let's look at the equations in vacuum i.e. $\vec{J} = 0$ and $\rho = 0$. Then $\nabla\cdot\vec{E} = 0$ and $\nabla\times\vec{H} = \frac{\partial \vec{E}}{\partial t}$.

$$\Rightarrow \nabla\times(\nabla\times\vec{E}) = \nabla\times\left(-\frac{\partial\vec{H}}{\partial t}\right) \overset{\text{check!}}{=} -\frac{\partial}{\partial t}(\nabla\times H)$$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\frac{\partial^2\vec{E}}{\partial t^2}.$$

But $\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$. (Exercise)

I) Four types of integrals.

1. Integral of scalar function f over curve c

   parametrizing $\vec{c}:[a,b] \to$ curve $\quad \int_{\vec{c}} f\, ds = \int_a^b f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt.$

   Relation to arclength: $\int_{\vec{c}} ds = \text{length(curve)} = \int_a^b \|\vec{c}\,'(t)\|\, dt.$

   Relation to average: $\int_{\vec{c}} f ds = \text{average}_c(f) \times \text{length}(c).$

2. Integral of scalar function f over surface S.

   parametrization: $\Phi: D \to S \quad \int_S f\, dS = \iint_D f(\Phi(u,v))\, \|T_u \times T_v\|\, du\, dv.$

   Relation to surface area: $\int_S dS = \iint_D \|T_u \times T_v\|\, du\, dv = A(S)$

   Relation to average: $\int_S f\, dS = \text{averages}_S(f) \times A(S).$

3. Integral of vector field $\vec{F}$ along oriented curve c (called "circulation" if c is closed).

   $\vec{c}: [a,b] \to$ curve, $\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt.$

4. Integral of vector field $\vec{F}$ ~~along~~ across an oriented surface S (called "flux").

   parametrization $\Phi: D \to S$, $\int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$

   $$= \iint_D \vec{F}(\Phi(u,v)) \cdot \vec{n}\, (\|T_u \times T_v\|)\, du\, dv$$

   where $\vec{n} = \dfrac{T_u \times T_v}{\|T_u \times T_v\|}.$

II. General advice on evaluating integrals.

1) Start by identifying which of the four types of integral (curve or surface; scalar or vector) you are asked to computed.

$$\int_C f\,ds\ ,\ \int_S f\,dS\ ,\ \int_C \vec{F}\cdot d\vec{s}\ ,\ \int_S \vec{F}\cdot d\vec{S}\ .$$

(I'll just talk about the two types of surface integral below, since the two types of curve integral are very similar).

2). The default method of evaluation is straightforward.

- Choose a parametrization $\Phi(u,v)$ of the surface S.
- Use the defining formula:

$$T_u \times T_v \quad \rightsquigarrow \quad \text{vector integral}$$
$$\|T_u \times T_v\| \quad\quad \text{scalar integral}$$

Important: be careful to distinguish during your calculations whether your integrals are over the original surface S or the parametrization domain D: They are completely different things!

3) Some standard types of parametrization are as follows

a. If the surface is the graph of some function $z = g(x,y)$, we can just take the parametrization to be $\Phi(u,v) = (u, v, g(u,v))$ and then speed things up using the formulae

$$T_u \times T_v = (-g_u, -g_v, 1) \quad \text{or} \quad \|T_u \times T_v\| = \sqrt{g_u^2 + g_v^2 + 1}$$

b. For part of a sphere of radius a, we can use the standard spherical coordinates given by Φ

$$\Phi(\theta, \phi) = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$ and use the standard formulae

$$T_\theta \times T_\phi = a^2\sin\phi\,(\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

or $\|T_\theta \times T_\phi\| = a^2\sin\phi$.

c. For part of a vertical cylinder of radius a, we can use the standard cylindrical coordinates given by

$\Phi(\theta, z) = (a\cos\theta,\ a\sin\theta,\ z)$ and use the formulae

$T_\theta \times T_z = (a\cos\theta,\ a\sin\theta,\ 0)$ or $\|T_\theta \times T_z\| = a$.

III. Useful tricks for evaluating integrals.

There's nothing wrong with the default method given above. It should always work as long as the functions involved in the integrand and the parametrization aren't too complicated - but it's not always the fastest method. We can rewrite integrals in various ways and sometimes one of those alternatives turns out to be easier to compute, so it's worth being aware of these possibilities.

1) integral you can calculate without integrating.

$$\iint_S 5\,dS = 5\,\text{Area}(S).$$

2) Change of variables, and/or reparametrization. Say you are doing an integral over a surface S which is the graph of a function $z = g(x,y)$ whose domain is a disc in the xy-plane. One you get down to an integral of the form $\iint_D f(x,y)\,dx\,dy$, you might want to change variables to polar coordinates in order to evaluate this.

3) The "right-to-left" direction can only be useful if you begin with an integral of the form $\int_C \vec{F}\cdot d\vec{s}$ where C is a closed oriented curve, because you have to be able to choose an oriented surface S whose boundary agrees with C. Because this direction increases the dimension of the integral, it tends to be useful only if $\nabla\times\vec{F}$ is a lot simpler than $\vec{F}$, or if a very ~~simple~~ nice surface S can be chosen (e.g. flat).

4) The same sorts of comments apply to using Gauss' thm. You need the divergence in the integrand to go left to right, and you have to start with a closed surface to go from right to left.

$$\iiint_W (\nabla\cdot\vec{F})\,dV \longleftrightarrow \iint_{\partial W} \vec{F}\cdot d\vec{S}.$$

5) Green's thm is simply the 2d version of Stoke's thm: for a vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$ and D a region in the plane

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} \vec{F}\cdot d\vec{s}.$$

To compute the area of D, choose

$Q = x$, $P = 0$ (or $Q = 0$, $P = -y$, etc.)

so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1, \text{ then } \int_{\partial D} \vec{F}\cdot d\vec{s} \text{ computes Area}(D).$$

(This is the same as if you actually start by writing down a parametrization of S using polar coordinates of the form $(r,\theta) \mapsto (r\cos\theta, r\sin\theta, g(r\cos\theta, r\sin\theta))$ and carrying out the computation using $T_r \times T_\theta$.

IV) Using Stoke's theorem and the other "fundamental theorems of calculus".

These are tricks which involve changing the dimension of the thing we're integrating over. Stoke's theorem can be used in either direction to replace one integral by the other:

$$\int_S (\nabla\times\vec{F})\cdot d\vec{S} \longleftrightarrow \int_{\partial S} \vec{F}\cdot d\vec{s}.$$

1) Make sure that you only try to use it in the "~~left to~~ left to right" direction when the integrand is the curl of something. Stoke's thm does not say that $\int_S \vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$!

This direction is typically useful because it reduces the dimension of the integral, which often makes evaluation easier.

2) A variation of this left-to-right direction is to change the surface S to a simpler surface S' with the same boundary as S, and then compute $\int_{S'}(\nabla\times\vec{F})\cdot d\vec{S}$ instead of $\int_S(\nabla\times\vec{F})\cdot d\vec{S}$; the thm tells us they are both equal to $\int_{\partial S}\vec{F}\cdot d\vec{s}$, so equal to one another. This trick might be useful if ∂S is a complicated curve, but there is a simple choice of surface S'.