- Path integrals:

parametrized curve $\vec{c}(t)$

scalar function $f$:

$$\int_{\vec{c}} f\,ds = \int_a^b \underbrace{f(\vec{c}(t))}_{\text{scalar}} \underbrace{\|\vec{c}'(t)\|}_{\text{scalar}}\,dt.$$

- Line integrals:

$\vec{F}$ vector field.

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}'(t)\,dt.$$

* Formulas for Surface Integrals:

1) Parametrized surface: $\Phi(u,v)$

a. $f$ scalar function.

$$\iint_S f\,dS = \iint_D \underbrace{f(\Phi(u,v))}_{\text{scalar}} \underbrace{\|T_u \times T_v\|}_{\text{scalar}}\,du\,dv$$

b. ~~Sca~~ Integral of a vector field $\vec{F}$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \underbrace{\vec{F}}_{\text{vector}}\cdot\underbrace{(\vec{T}_u\times\vec{T}_v)}_{\text{vector}}\,du\,dv$$

2) Graph: $z = g(x,y)$

a) $f$ scalar

$$\iint f\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dx\,dy$$

$$= \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy.$$

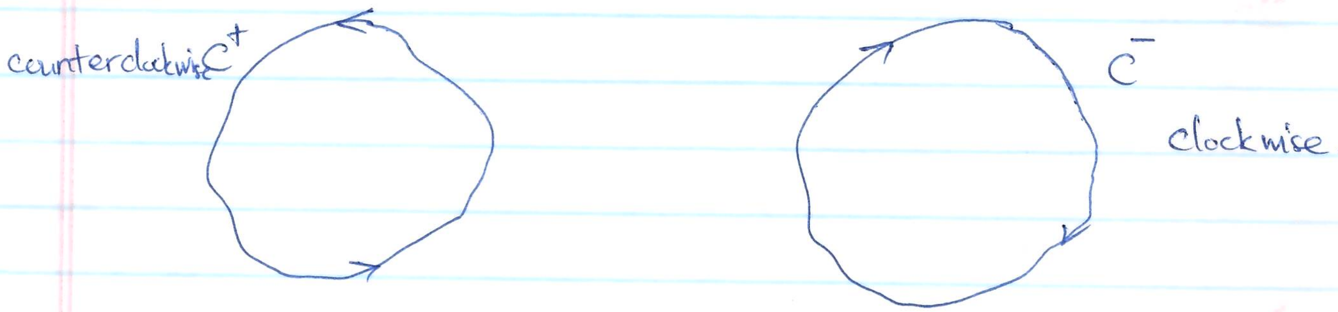
where $\cos\theta = \vec{n}\cdot\vec{k}$, and $\vec{n}$ is a unit normal vector to the surface.

b) $\vec{F} = (F_1, F_2, F_3)$ vector field.

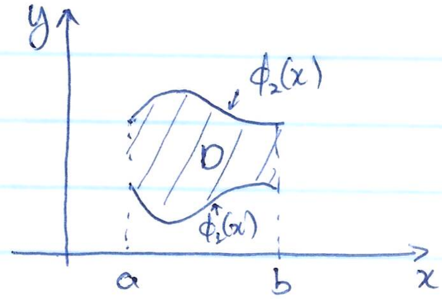
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-F_1 g_x - F_2 g_y + F_3)\,dx\,dy$$

## 8.1. Green's Theorem.

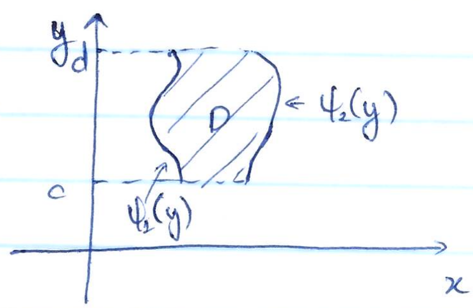
Relates line integral along a closed curve $C$ in $\mathbb{R}^2$ to a double integral over a region enclosed by $C$.



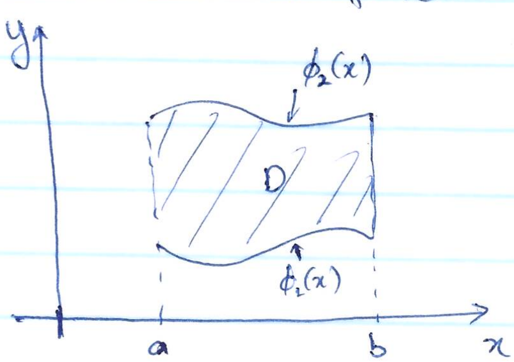

Recall:
- $y$-simple region
  $a \le x \le b$
  $\phi_1(x) \le y \le \phi_2(x)$




- $x$-simple region
  $c \le y \le d$
  $\psi_1(y) \le x \le \psi_2(y)$.




Take a $y$-simple region $D$, with boundary $C$ and let $P: D \to \mathbb{R}$ be a function



$$\iint_D \frac{\partial P}{\partial y}(x,y)\,dxdy = \underbrace{\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy}_{\text{FTOC}}\,dx$$

$$= \int_a^b P(x,\phi_2(x)) - P(x,\phi_1(x))\,dx.$$

$(x, \phi_2(x))$, for $a \le x \le b$, is the parametrization of the top part of $C$ going from $a$ to $b$.
and $(x, \phi_1(x))$, for $a \le x \le b$, is the parametrization of the bottom part of $C$ going from $a$ to $b$.

Then

$$\int_a^b P(x,\phi_2(x))\,dx = \int_{C(\text{top}, a\to b)} P(x,y)\,dx. \qquad (\mathrm{I})$$

and

$$\int_a^b P(x,\phi_1(x))\,dx = \int_{C(\text{bottom}, a\to b)} P(x,y)\,dx. \qquad (\mathrm{II})$$

$$= -\int_{C(\text{bottom}, b\to a)} P(x,y)\,dx.$$

In addition,

$$\int_{C(\text{left})} P(x,y)\,dx = \int_{C(\text{right})} P(x,y)\,dx = 0 \qquad (\mathrm{III})$$

as $x$ is constant.

$$\Rightarrow \boxed{\iint_D \frac{\partial P}{\partial y}\,dxdy = \int_{C^-} P\,dx = -\int_{C^+} P\,dx.}$$

for a $y$-simple region $D$.

Similarly, for a $x$-simple region $D$

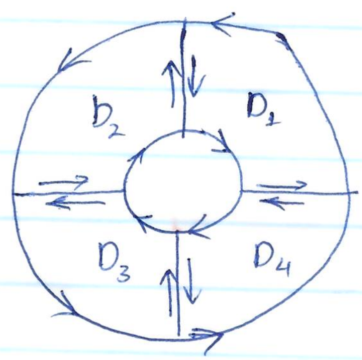
$$\boxed{\iint_D \frac{\partial Q}{\partial x}\,dxdy = \int_{C^+} Q\,dy = -\int_{C^-} Q\,dy}$$

* <u>Green's Theorem:</u> Let $D$ be a simple region (i.e. both ~~x-y~~ $x$-simple and $y$-simple) and let $C$ be its boundary. Let $P: D \subset \mathbb{R}^2 \to \mathbb{R}$ and $Q: D \subset \mathbb{R}^2 \to \mathbb{R}$ have continuous partials.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\underset{\uparrow}{C^+}} P\,dx + Q\,dy.$$

counterclockwise.

<u>Remark:</u> If the region is not simple, we can break it up into simple regions and sum them up.




<u>Remark:</u> If you walk along the curve $C$ with the correct orientation, the region $D$ will be on your left.

<u>E.g.</u> Evaluate using Green's theorem.

$$\int_{C^+} \underbrace{y^3}_{P} dx \underbrace{- x^3}_{Q} dy \qquad \text{where } C \text{ is the unit circle.}$$

<u>Sol:</u>

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{D \leftarrow \text{unit disk}} (-3x^2 - 3y^2)\, dx\,dy.$$

$$= -3 \iint_D (x^2 + y^2)\, dx\,dy$$

$$\overset{\text{polar coordinates}}{=} \int_0^{2\pi}\int_0^1 -3r^2 \, r\,dr\,d\theta$$

$$= 2\pi \left(-\frac{3r^4}{4}\right)_0^1$$

$$= \cancel{\frac{6\pi}{8}} \ldots -\frac{6\pi}{4} = -\frac{3\pi}{2}.$$

Verify by evaluating $\int_{C^+} y^3dx - x^3dy$ directly.

$$\int_{C^+} y^3 dx - x^3 dy = \cancel{\int_0^{2\pi} (\cos^3\theta)\cdot(\sin^3\theta)(-\sin}$$

$$= \int_0^{2\pi} (\sin^3\theta)(-\sin\theta) - (\cos^3\theta)(\cos\theta)\, d\theta.$$

$$= \int_0^{2\pi} -(\sin^4\theta + \cos^4\theta)\, d\theta$$

$$= \ldots = -\frac{6\pi}{4}.$$

~~integration by parts~~ / trig. identities.

* Important applications of Green's Theorem.

$$\iint_D dx\,dy = A(D) = \frac{1}{2}\iint_{\partial D} x\,dy - y\,dx.$$

$\partial D$: boundary of $D$, oriented counterclockwise.

Pf: $$\frac{1}{2}\iint_{\partial D} \underbrace{x}_{Q}\,dy - \underbrace{y}_{P}\,dx = \frac{1}{2}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

$$= \frac{1}{2}\iint_D (1+1)\, dx\,dy$$

$$= \iint_D dx\,dy.$$

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E.g. Compute the area of the hypercycloid
$S: (a\cos^3\theta,\ a\sin^3\theta)$ , $0 \le \theta \le 2\pi$.

$$A = \frac{1}{2}\iint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos^3\theta\cdot(a\,3\sin^2\theta\cos\theta) - a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 3a^2\cos^4\theta\sin^2\theta + 3a^2\sin^4\theta\cos^2\theta\,d\theta.$$

$$= \frac{1}{2}\int_0^{2\pi} 3a^2\cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta.$$

$$= \frac{3a^2}{8}\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta \qquad \text{since } (\sin t)^2 = \frac{1-\cos 2t}{2}.$$

$$= \frac{3a^2}{8}\pi.$$

* Curl of a vector field.

Del operator $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

$f: \mathbb{R}^3 \to \mathbb{R}$

then gradient $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$.

curl: $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

where $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$
$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Eg. Compute the curl of $\vec{F} = y\vec{i} - x\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \frac{\partial x}{\partial z}\vec{i} - \frac{\partial y}{\partial z}\vec{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right)\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + (-2)\vec{k}.$$

* Vector form of Green's theorem.

Suppose $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$

and suppose we have a nice region $D$ with boundary $\partial D$ (positively oriented).

Then $$\int_{\partial D} P\,dx + Q\,dy = \int_{\partial D} \vec{F}\cdot d\vec{s}.$$

Moreover, $$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_D (\nabla \times \vec{F})\cdot\vec{k}\,dx\,dy.$$

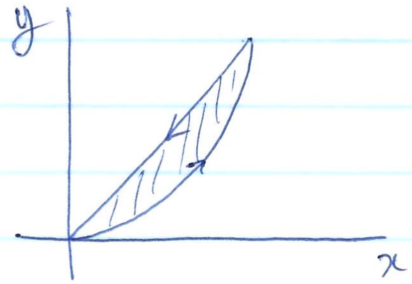
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

$$\Rightarrow (\nabla \times \vec{F})\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

$$\Rightarrow \boxed{\iint_{\partial D} \vec{F}\cdot d\vec{s} = \iint_D (\nabla \times \vec{F})\cdot\vec{k}\,dx\,dy.}$$

vector form of Green's Theorem.

<u>E.g.</u> let $\vec{F} = (xy, x-y)$.
Compute $\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx dy$ where $D$ is the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$




By Green's theorem, $\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx dy = \iint_{\partial D} \vec{F} \cdot d\vec{s}$.

1) Along the parabola from $x = 0$ to $x = 1$.

$\vec{c}(x) = (x, x^2)$

$$\int_0^1 (F_1, F_2) \cdot (1, 2x)\, dx = \int_0^1 (x^3, x - x^2) \cdot (1, 2x)\, dx$$

$$= \int_0^1 x^3 + 2x^2 - 2x^3\, dx.$$

$$= \int_0^1 2x^2 - x^3\, dx$$

$$= \frac{2x^3}{3} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

2) Along the line $y = x$ from $x = 1$ to $x = 0$.

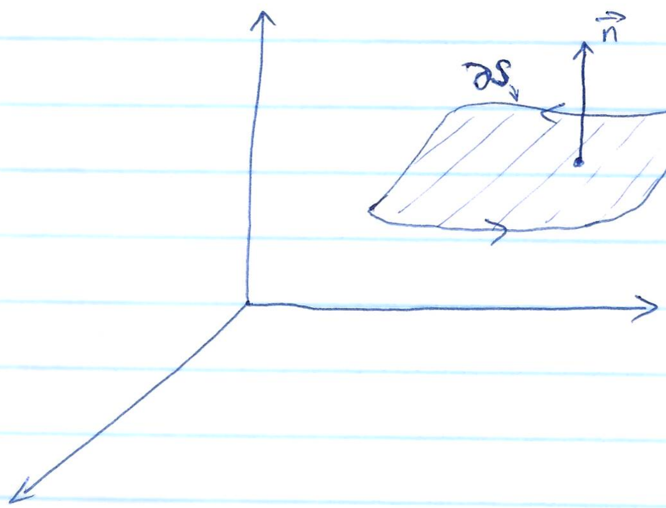
$\vec{c}(x) = (x, x)$.

$$\int_1^0 (x^2, 0) \cdot (1, 1)\, dx = \int_1^0 x^2\, dx = \frac{x^3}{3}\Big|_1^0 = -\frac{1}{3}.$$

$$\Rightarrow \int_{\partial D} \vec{F} \cdot d\vec{s} = \frac{5}{12} - \frac{1}{3} = \frac{1}{12} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx dy.$$

## Stoke's Theorem

- Green's theorem: dealt with planar (flat) regions.
- Stoke's thm: relates the line integrals of a vector field around a simple closed curve C to an integral over a surface S with C as its boundary.

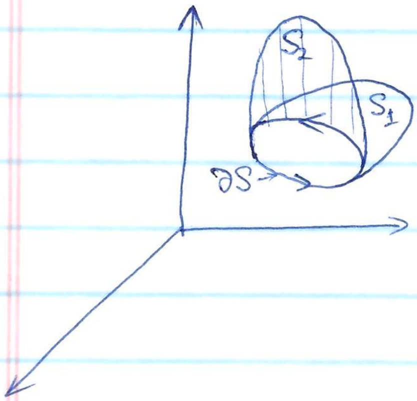



Stoke's Thm: If C is a closed curve in space, and S is a surface bounded by C, then

$$\int_{\vec{C}} \vec{F}\cdot d\vec{s} = \iint_{S} (\nabla\times\vec{F})\cdot d\vec{S}.$$

Rule to determine orientation: walking along C in the + direction, with S ~~to~~ your left, $\vec{n}$ should be pointing up.

Observation: It doesn't matter what S is as long as it's boundary is C.




$S_1$ & $S_2$ have the same boundary ?
so $\displaystyle\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_{S_1} (\nabla\times\vec{F})\cdot d\vec{S}_1$

$\displaystyle = \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S}_2$ .

<u>E.g.</u> Verify Stoke's theorem for
$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, $C =$ unit circle in the $xy$-plane oriented counterclockwise.
$(\cos\theta, \sin\theta, 0)$.

$S: z = 1 - x^2 - y^2$.

$$\int_{C^+} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}\cdot\vec{c}\,'(\theta)\, d\theta = \int_0^{2\pi} (0, \cos\theta, \sin\theta)\cdot \overset{-\sin\theta,\ \cos\theta,\ 0}{(\cancel{-2\cos\theta, -2\sin\theta, 0})}\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta.$$

$$= \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta.$$

$$= \pi.$$

On the other hand,
$\displaystyle\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = ?$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}.$$

Since $S: (x, y, \underbrace{1-x^2-y^2}_{g})$

then $$\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) = (2x, 2y, 1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$$

$$\text{So } \iint_S (\nabla\times\vec{F})\cdot \overbrace{\overrightarrow{dS}}^{(\vec{T}_x\times\vec{T}_y)dxdy} = \iint_{\text{unit disk}} 2x + 2y + 1 \, dxdy.$$

$$= \int_0^1 \int_0^{2\pi} (2r\cos\theta + 2r\sin\theta + 1)\, r \, d\theta \, dr$$

$$= \int_0^1 0 + 0 + 2\pi r \, dr$$

$$= \pi.$$

Eg. Let $S$ be a surface whose boundary is the circle $x^2 + y^2 = 1$, where $S$ lies above the $xy$ plane with normal vector having positive $\vec{k}$ component.

Let $\vec{F} = y\vec{i} - x\vec{j} + e^{xz}\vec{k}$ and compute $\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$.

Sol: By Stoke's theorem $\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\vec{c}} \vec{F}\cdot d\vec{s}$

where $C: (\cos\theta, \sin\theta, 0)$ and $0 \le \theta \le 2\pi$

$\Rightarrow$ oriented counterclockwise.

$$\Rightarrow \int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \underbrace{(\sin\theta, -\cos\theta, e^0)}_{\vec{F}(\vec{c}(\theta))} \cdot \underbrace{(-\sin\theta, \cos\theta, 0)}_{\vec{c}\,'(\theta)} \, d\theta$$

$$= \int_0^{2\pi} -1 \, d\theta$$

$$= -2\pi$$

Remark: • we didn't care what $S$ was explicitly.

• (Important) For a surface $S$ with no boundary (e.g. sphere) $\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = 0$ by Stoke's theorem.