7.2 Line Integrals.

Motivation

$$W = (\text{force}) \cdot \cancel{\text{distance}}\ (\text{displacement in the direction of force}).$$

over the short distance: $W \approx \vec{F} \cdot \vec{\Delta s}$

total work over long distance along trajectory c:

$$W \approx \sum_i \vec{F}(t_i) \cdot (\Delta \vec{s})_i \longrightarrow \int_{\vec{c}} \vec{F} \cdot d\vec{s}.$$

$$\Rightarrow \quad W = \int_{\vec{c}} \vec{F} \cdot d\vec{s}.$$

Def: Let $\vec{F}$ be a vector field in $\mathbb{R}^3$, continuous on $\vec{c}: [a, b] \to \mathbb{R}^3$. The line integral of $\vec{F}$ along $\vec{c}$ is defined as

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int \vec{F}(x(t), y(t), z(t)) \cdot \vec{c}'(t)\, dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t), \quad 0 \le t \le 2\pi.$

$\vec{F}(x, y, z) = x^2 \vec{i} + y^2 \vec{j} + z\vec{k}.$

Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

Sol: Along $\vec{c}$,

$$\vec{F}(x, y, z) = \vec{F}(\cos t, \sin t, t) = \cos^2 t\, \vec{i} + \sin^2 t\, \vec{j} + t\, \vec{k}.$$

~~then~~ and

$$\vec{c}'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + 1\vec{k}.$$

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$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (\cos^2 t\, \vec{i} + \sin^2 t\, \vec{j} + t\vec{k}) \cdot (-\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k})\, dt$$

$$= \int_0^{2\pi} -\cos^2 t \sin t + \sin^2 t \cos t + t\, dt.$$

$$= \int_0^{2\pi} -\cos^2 t \sin t\, dt + \int_0^{2\pi} \sin^2 t \cos t\, dt + \int_0^{2\pi} t\, dt$$

$$= \left.\frac{\cos^3 t}{3}\right|_0^{2\pi} + \left.\frac{\sin^3 t}{3}\right|_0^{2\pi} + \left.\frac{t^2}{2}\right|_0^{2\pi}$$

$$= \cancel{\cos^3}\ 0 + 0 + \frac{4\pi^2}{2}$$

$$= 2\pi^2.$$

New notation: for the line integral

$\vec{F} = (P, Q, R)$

and $d\vec{s} = (dx, dy, dz)$

We write

$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

Remark: this is not the sum of 3 integrals. It is just another way to write $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$. We still need to parametrize and express everything in terms of the parameter.

E.g. $\vec{c}(t) = (\cos t, \sin t, t) \qquad 0 \leq t \leq 2\pi$

$\vec{F}(x,y,z) = \underset{\underset{P}{\uparrow}}{x^2}\vec{i} + \underset{\underset{Q}{\uparrow}}{y^2}\vec{j} + \underset{\underset{R}{\uparrow}}{z}\vec{k}.$

so $\displaystyle\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{\vec{c}} x^2 dx + \oint y^2 dy + z dz.$

$\displaystyle = \int_a^{2\pi} \left(x^2(t)\frac{dx}{dt} + y^2(t)\frac{dy}{dt} + z(t)\frac{dz}{dt} \right) dt$

$\displaystyle = \int_0^{2\pi} (\cos^2 t(-\sin t) + \sin^2 t \cos t + t)\, dt.$

$= 2\pi^2.$

E.g. Evaluate $\displaystyle\int_{\vec{c}} x^2 dx + xy\,dy$

where $\vec{c}(t) = (t, t^2), \quad 0 \leq t \leq 1.$

$\displaystyle\int_{\vec{c}} x^2 dx + xy\,dy = \int_0^1 \left(x^2(t)\frac{dx}{dt} + x(t)\,y(t)\frac{dy}{dt} \right) dt.$

$\displaystyle = \int_0^1 (t^2 + t\cdot t^2 \cdot 2t)\, dt$

$\displaystyle = \int_0^1 (t^2 + 2t^4)\, dt$

$\displaystyle = \frac{t^3}{3} + \frac{2}{5}t^5 \Big|_0^1$

$= \frac{1}{3} + \frac{2}{5}.$

$= \frac{11}{15}.$

Remark: line integrals are independent of the parametrization as long as the parametrization is orientation preserving.

E.g. The curve $y = x^3$ from $(0,0)$ to $(1,1)$ can be parametrized as $\vec{c}(t) = (t, t^3)$ $\quad 0 \leq t \leq 1$

or as $\vec{p}(\theta) = (\sin\theta, \sin^3\theta)$ $\quad 0 \leq \theta \leq \pi/2$.

Let $\vec{F} = x\vec{i} + y\vec{j}$.

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 (t\vec{i} + t^3\vec{j}) \cdot \vec{c}\,'(t)\, dt$$

$$\text{or} \quad = \int_0^{\pi/2} (\sin\theta\,\vec{i} + \sin^3\theta\,\vec{j}) \cdot \vec{p}\,'(\theta)\, d\theta.$$

Check!

* Line integral of gradient field:

FTOC: $\int_a^b f'(t)\, dt = f(b) - f(a)$.

Thm: (Fundamental theorem of line integrals).

$f: \mathbb{R}^3 \to \mathbb{R}$ differentiable and $c: [a,b] \to \mathbb{R}$.

(c is continuous or piecewise ~~cts~~ cts.)

$$\int_{\vec{c}} (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

Remark: If the field is a gradient field, only the end points matter.

E.g. Evaluate $\int_{\vec{c}} \nabla f \cdot d\vec{s}$ where ~~$f(x, y, z)$~~

$$f(x,y,z) = \cos x + \sin y - xyz$$

and $\vec{c}$ is a trajectory that starts at $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ and ends at $(\pi, 2\pi, 1)$.

Sol: Let $\vec{c}(t)$, $a \le t \le b$, be a path with $\vec{c}(a) = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ & $\vec{c}(b) = \left(\pi, \frac{2\pi}{2}, 1\right)$.

Then by ~~FTOLI~~ FTOLI.

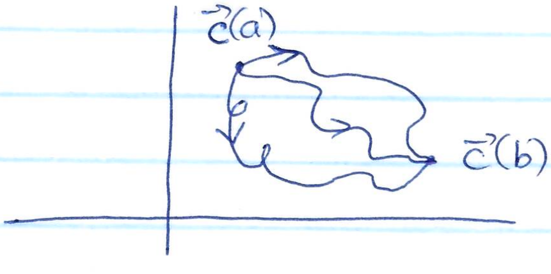
$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

$$= f\left(\pi, \frac{2\pi}{2}, 1\right) - f\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$$

$$= \cos\pi + \sin\frac{2\pi}{2} - \frac{\pi^2}{2} - \cos\frac{\pi}{2} - \sin\frac{\pi}{2}$$

$$= -1 + 0 - \frac{\pi^2}{2} - 0 - 1$$

$$= -2 - \frac{\pi^2}{2}.$$



- Integrals of <u>scalar fields</u> along curves
  $\Rightarrow$ path integrals

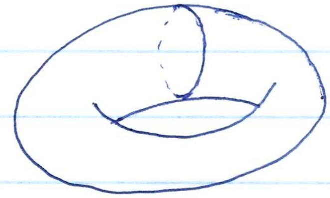
- Integrals of <u>vector fields</u> along curves
  $\Rightarrow$ line integrals.

7.3 Parametrized Surfaces.

E.g. The graph of a function $f(x,y)$ is a surface.

But we still have surfaces that are not the graph of a function.

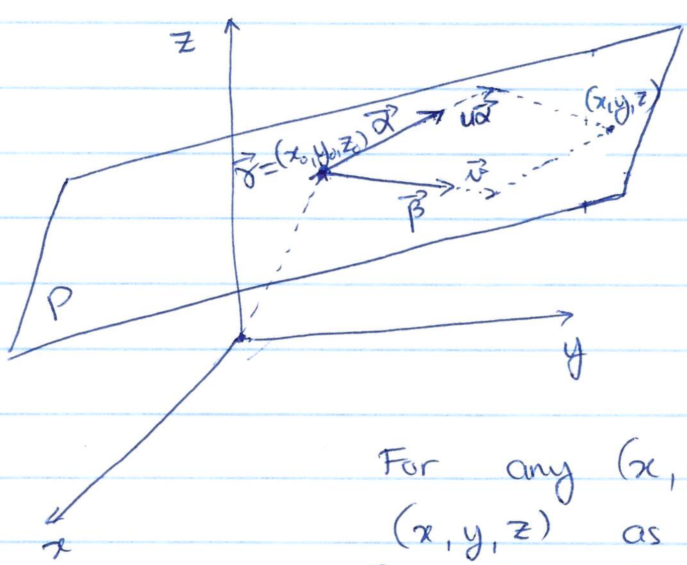
E.g. Torus (surface of a donut).



> **Def:** A parametrization of a surface is a function $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$. The surface is $S = \Phi(D)$ and
> $$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)).$$

If $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable, we call S a differentiable surface.

- Parametrization of a plane:



Let P be a plane that is parallel to $\vec{\alpha}$ and $\vec{\beta}$, and passes through $\vec{\gamma}$.

For any $(x,y,z) \in P$, we can write (x,y,z) as

$$(x,y,z) = (x_0,y_0,z_0) + u\vec{\alpha} + v\vec{\beta}$$

for some $u, v \in \mathbb{R}$

So $\boxed{\Phi(u,v) = \vec{\alpha} u + \vec{\beta} v + \vec{\gamma}}$

E.g. Find a parametrization of the plane $x+y+z=1$

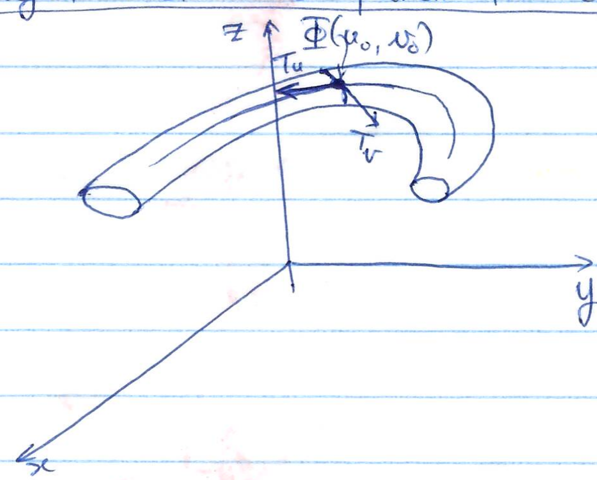
Sol: The point $(0,0,1)$ is on the plane.

Vectors $(1,-1,0)$ and $(0,1,-1)$ are parallel to the plane (Why?)

Hence,

$$\Phi(u,v) = (1,-1,0)u + (0,1,-1)v + (0,0,1).$$

* Tangent vectors to parametrized surfaces:



Suppose that $\Phi(u,v)$ is diff. at (u_0, v_0).

Fix v_0 and look at the map $t \mapsto \Phi(u_0, t, v_0)$ (in other words, we have a map $\mathbb{R} \to \mathbb{R}^3$) which identifies a ~~so~~ curve on the surface.

The vector tangent to this curve at (u_0, v_0) is given by

$$T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\vec{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\vec{k}.$$

It is also tangent to the surface.

Similarly, $T_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k}$

Both T_u and T_v are tangent to the surface.
so $T_u \times T_v$ is normal to them (provided $T_u \times T_v \neq 0$).

To find the equation of the tangent plane at (u_0, v_0)
we calculate

$$\vec{\eta} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}_{(u_0, v_0)}$$

Tangent plane is

$$\vec{\eta} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

$$\Leftrightarrow \quad n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \qquad \vec{n} = (n_1, n_2, n_3)$$

E.g. Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\Phi(u, v) = (u\cos v, u\sin v, u^2 + v^2)$$

Find the tangent plane at $\Phi(1, 0)$.

Sol: $T_u = (\cos v, \sin v, 2u)$

$T_v = (-u\sin v, u\cos v, 2v)$

$$\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 2v \end{vmatrix}_{(1,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= -2\vec{i} - (0)\vec{j} + 1\cdot\vec{k}$$

$$= -2\vec{i} + \vec{k}.$$

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$(x_0, y_0, z_0) = \Phi(u_0, v_0) = \Phi(1,0) = (1, 0, 1).$

The eq. of the tangent plane

$$-2(x-1) + 0(y-0) + 1(z-1) = 0$$

$$-2x + 2 + z - 1 = 0.$$

$$-2x + z = -1.$$

Remark: We say that a surface is regular or smooth, at $\Phi(u_0, v_0)$ if $T_u \times T_v \neq 0$ at (u_0, v_0). We say that it is regular if it is regular at all points $\Phi(u_0, v_0) \in S$.

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7.4 Area of a surface.

Def: The surface area of a parametrized surface

$$A(S) = \iint_D \|T_u \times T_v\| \, du\,dv.$$

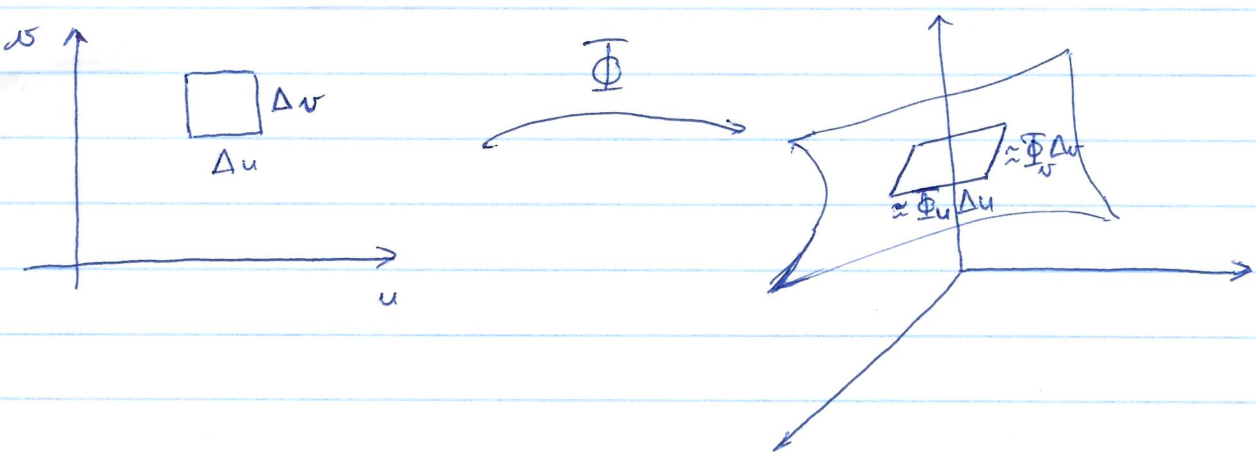
S is regular

Φ one-to-one and differentiable.

Recall: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, then

$$\|T_u \times T_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}$$

So $$\boxed{A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du\,dv}$$



The area of the image of the small rectangle is

$$\|(\Phi_u \Delta u) \times (\Phi_v \Delta v)\| = \|\Phi_u \times \Phi_v\| \, \Delta u \, \Delta v.$$

Summing the areas of all small rectangles as $\Delta u, \Delta v \to 0$ gives $A(S)$.

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E.g. Find the surface area of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1.$$

Parametrize:

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1, \; 0 \le \theta \le 2\pi.$$

Then $T_r = (\cos\theta, \sin\theta, 1)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$$T_r \times T_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\,\vec{i} \mp r\sin\theta\,\vec{j} + r\vec{k}$$

$$\|T_r \times T_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2}$$

$$= \sqrt{2r^2}$$

$$= \sqrt{2}\, r$$

$$A(\text{cone}) = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\,dr\,d\theta = 2\pi\sqrt{2}\,\frac{r^2}{2}\Big|_0^1 = \sqrt{2}\,\pi.$$

* Surface area of the graph of a function $f(x,y)$.

We can parametrize S by $(x, y, f(x,y))$

or $(u, v, f(u,v))$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = -f_u\,\vec{i} - f_v\,\vec{j} + 1\vec{k}.$$

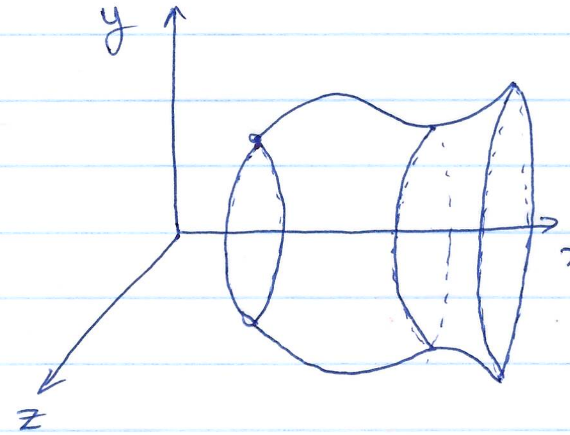
$$A(S) = \iint_D \sqrt{f_u^2 + f_v^2 + 1}\;du\,dv.$$

Exercise: use this to find the area of the cone

$$z = \sqrt{x^2 + y^2}$$

* <u>Surfaces of revolution</u>.

Suppose S is obtained by rotating the graph of $y = f(x)$, $a \le x \le b$ around the x-axis.



We can parametrized S as $\quad (u, f(u)\cos v, f(u)\sin v)$

$$a \le u \le b, \quad 0 \le v \le 2\pi.$$

So
$$T_u = (1, f'(u)\cos v, f'(u)\sin v)$$
$$T_v = (0, -f(u)\sin v, f(u)\cos v)$$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix}$$

$$= f'(u) f(u)\,\vec{i} - f(u)\cos v\,\vec{j} - f(u)\sin v\,\vec{k}.$$

$$\Rightarrow A(S) = \int_0^{2\pi}\int_a^b \sqrt{f'(u)^2 + \cos^2 v + \sin^2 v}\;|f(u)|\,du\,dv.$$

$$= \int_a^b\int_0^{2\pi} \sqrt{f'(u)^2 + 1}\;|f(u)|\,dv\,du.$$

$$= 2\pi \int_a^b \sqrt{f'(u)^2 + 1}\;|f(u)|\,du.$$

<u>Exercise</u>: use this to compute the area of the cone $z = \sqrt{x^2+y^2}$. $\quad (0 \le x \le 1)$.