E.g. Calculate the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$

Sol: $A(D) = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dx\,dy.$

$u = \frac{x}{a}$

$v = \frac{y}{b}$.

$A(D) = \iint\limits_{u^2+v^2\le 1} ab\,du\,dv$

$= ab \iint\limits_{u^2+v^2\le 1} du\,dv$

$= \pi ab$.

Solve: $\int_0^1 xe^{x^2}\,dx \rightarrow$ change of variable/ substitution.

Let $u = x^2 \Rightarrow du = 2x\,dx$.

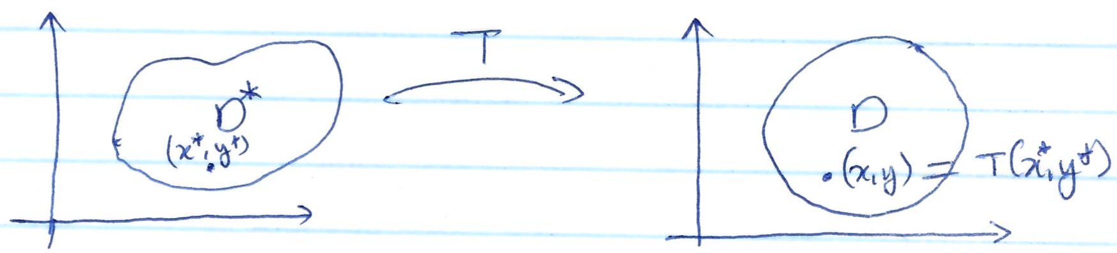
$$\int_0^1 xe^{x^2}\,dx = \int_0^1 \frac{e^u}{2}\,du = \frac{e^u}{2}\Big|_0^1 = \frac{e}{2} - \frac{1}{2}.$$

When we have several variables, we also need to do something similar. In this chapter, we develop the multidimensional change of variables formula, ~~which~~ -i

6.1. The geometry of maps from $\mathbb{R}^2 \to \mathbb{R}^2$.

Let T be a map from $D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$

We call $D = T(D^*)$ the set of image points of T (so every point (x,y) in D must be equal to $T(x^*,y^*)$ for some (x^*,y^*) in D^*).



Eg. (Polar coordinates)

Let $D^* \subset \mathbb{R}^2$ be the rectangle $D^* = [0,1] \times [0,2\pi]$, i.e. all points in D^* are of the form (r,θ), where $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

$$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}).$$

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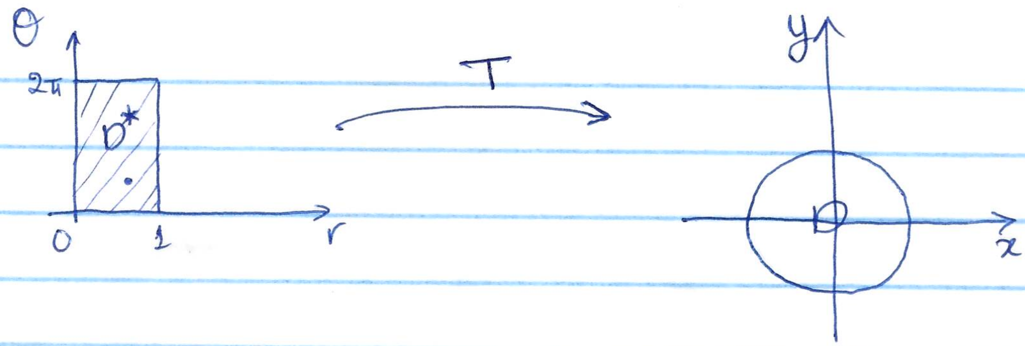

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2 \leq 1.$$

$\Rightarrow$ $D = T(D^*)$ is contained in a unit disk.

<u>Q</u>: Is it the whole unit disk? Yes!

Because for any (x, y) in the unit disk, (x, y) can be written as $(r\cos\theta, r\sin\theta)$ for some $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

<u>Thm</u>: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix with $\det(A) = ad - bc \neq 0$.
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = A\vec{x}$.
(in other words, $T(x, y) = (ax + by, cx + dy)$).

Then T transforms parallelogram into parallelogram and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

<u>E.g.</u> Let $T(x, y) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}$.

and let $D^* = [-1, 1] \times [-1, 1]$.

Find D?

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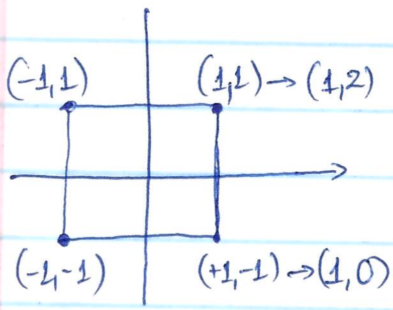
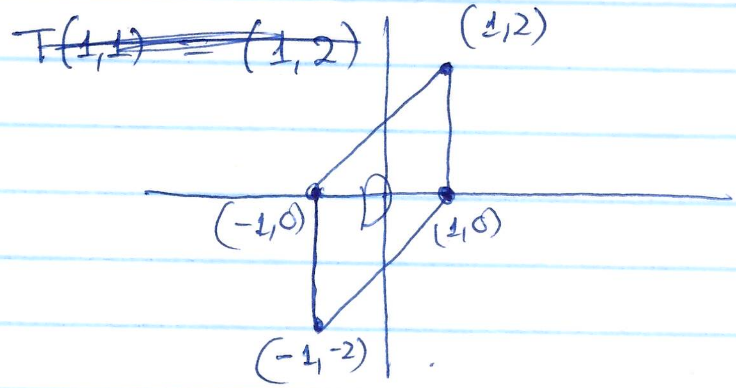
$$T(x,y) = \begin{pmatrix} x \\ x+y \end{pmatrix}$$

Sol. $\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 \neq 0$. And D^* is a rectangle.

$\Rightarrow$ we only need to map the vertices and connect them.





$T(1,1) = (1,2)$

$T(1,-1) = (1,0)$

$T(-1,1) = (-1,0)$

$T(-1,-1) = (-1,-2)$

Def: A mapping T is one-to-one on D^* if for (u,v) and $(u',v') \in D^*$,

$$T(u,v) = T(u',v') \Rightarrow u = u',\ v = v'.$$

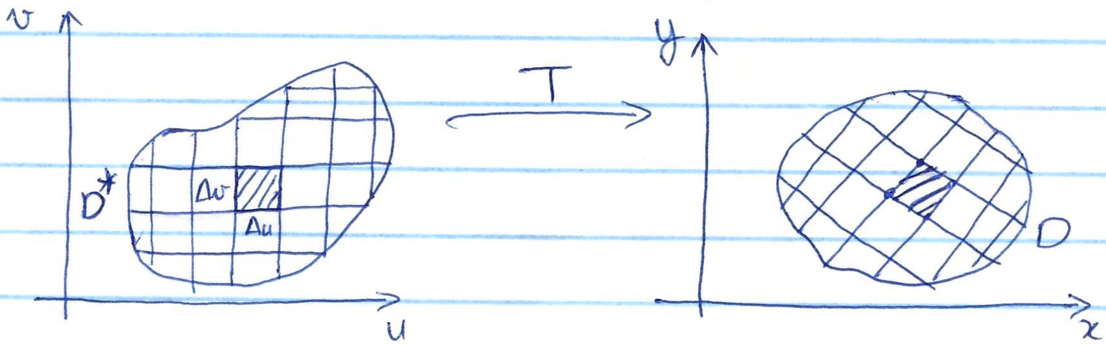
* 6.2. Change of Variables Theorem.

Want to evaluate $\iint_D f(x,y)\,dx\,dy$

but too hard in some cases.

$\Rightarrow$ use a chage of variable.

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(u,v) = (x(u,v),\ y(u,v)).$$



Find the area

$$\text{AreA}(D) = \iint_D dx\,dy.$$

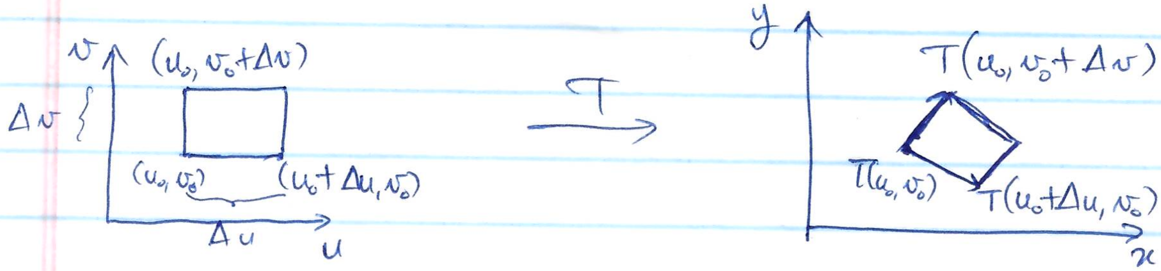
Area(D) = sum of areas of the little "almost" ~~paral~~ parallelogram.

$\approx$ approximately sum of areas of parallelograms obtained by linear approximation of T.

Taking limit as the parallelograms become very small,

$$\text{Area}(D) = \iint_D dx\,dy.$$

$\Rightarrow$ we need the areas of the parallelograms.



linear approximation:

$$T(u,v) \approx T(u_0, v_0) + T'\begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}.$$

$$\Rightarrow T(u_0+\Delta u, v_0) \approx T(u_0, v_0) + T'\begin{pmatrix} \Delta u \\ 0 \end{pmatrix}$$

$$= T(u_0, v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{pmatrix} \Delta u \\ 0 \end{pmatrix}$$

Similarly,

$$T(u_0, v_0+\Delta v) \approx T(u_0, v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{pmatrix} 0 \\ \Delta v \end{pmatrix}.$$

$\Rightarrow$ the sides of the parallelogram are

$$T(u_0+\Delta u, v_0) - T(u_0, v_0) = \begin{pmatrix} \frac{\partial x}{\partial u}\cdot\Delta u \\ \frac{\partial y}{\partial u}\Delta u \end{pmatrix}$$

and

$$T(u_0, v_0+\Delta v) - T(u_0, v_0) = \begin{pmatrix} \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial v}\Delta v \end{pmatrix}.$$

$\Rightarrow$ The area of the parallelogram.

$$\begin{vmatrix} \frac{\partial x}{\partial u}\cdot\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\cdot\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} \leftarrow \text{determinant.}$$

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$$= \left| \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right) \Delta u \Delta v \right|$$

$$= \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| \Delta u \Delta v .$$

$$\Rightarrow \text{Area}(D) \approx \sum_{\text{all } (u_0, v_0)} \sum \text{ areas of the parallelogram}$$

$$= \sum_{\text{all } (u_0, v_0)} \sum \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|_{(u_0, v_0)} \Delta u \Delta v .$$

$$\xrightarrow[\text{shrinks}]{\text{as parallelogram}} \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| \cancel{dx\,dy} \cdot du\,dv$$

$$\Rightarrow \text{Area}(D) = \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\,dv .$$

In general, the change of variable formula is

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \cancel{d} \underbrace{\left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|}_{\substack{\text{Jacobian} \\ \text{determinant}}} du\,dv$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| .$$

(27)

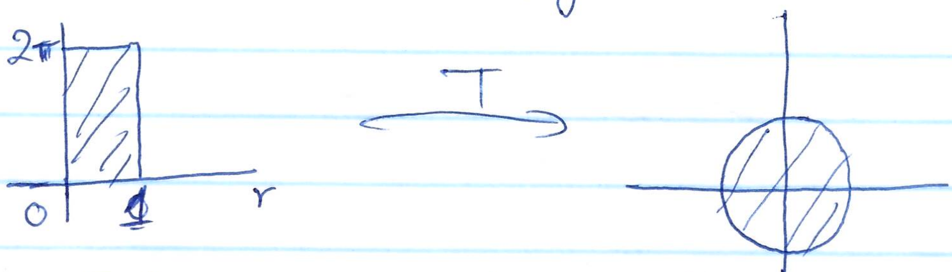
E.g. Use polar coordinates to find

$$\underbrace{\iint_{x^2+y^2\le 1}}_{D} e^{-(x^2+y^2)}\,dx\,dy.$$

Sol: $x = r\cos\theta$

$y = r\sin\theta.$

$T(r,\theta) = (x, y)$



Find Jacobian determinant.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin\theta)\sin\theta$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

$$\cancel{\iint_D e^{-r^2}}\, r\,dr \Rightarrow \iint_D e^{-(x^2+y^2)}\,dx\,dy = \iint_{D^*} e^{-r^2}\, r\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^{1} r e^{-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1} r e^{-r^2}\,dr$$

$$= 2\pi \cdot \left(-e^{-r^2}\right)\Big|_0^1$$

$$= 2\pi\left(1 - e^{-1}\right).$$

In general, polar coordinates change of variable formu

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

---

* Change of variables formula for triple integrals.

Let T be a function from $\mathbb{R}^3 \to \mathbb{R}^3$.

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

$\Rightarrow$ The Jacobian determinant of T:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Two important cases

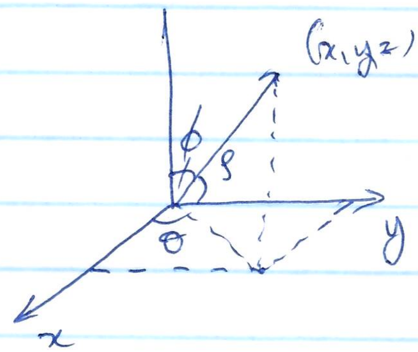
1) Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$$\rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

2) ~~Sph~~ Spherical coordinates:

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\,\cancel{\cos\theta\cos\phi}\ \sin\theta\sin\phi.$$
$$z = \rho\cos\phi.$$



$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \dots = \rho^2\sin\phi.$$

$$\Rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \cancel{\iiint_{W^*} f(x(\rho,\theta,\phi), y(\rho,\theta,\phi))}.$$

$$= \iiint_{W^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

(30).

E.g. Compute the volume of the solid W between the paraboloids: $z = x^2 + y^2$ and $z = 1 - x^2 - y^2$.

* 4.3 Vector Fields:

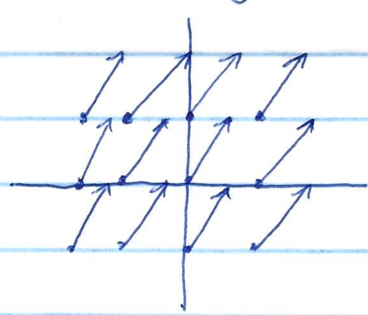
A vector field $\vec{F}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that assigns to each point $\vec{x} = (x_1, x_2, \ldots, x_n)$ a vector $\vec{F}(\vec{x}) = \vec{F}(x_1, \ldots, x_n)$.

E.g. velocity fields (wind, fluids)
force fields (magnetic, gravitational).

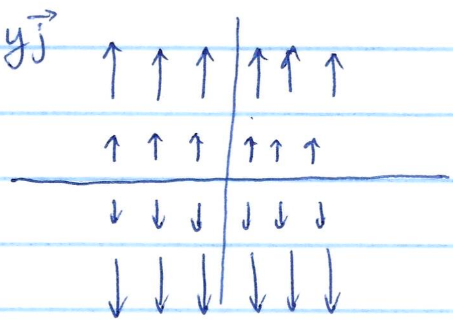
in 2D: $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$.

in 3D: $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$.

E.g. $\vec{F} = \vec{i} + \vec{j}$



$\vec{F} = y\vec{j}$



* Gradient vector fields:

Given a function $f(x,y,z)$, its gradient is

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

$\Rightarrow$ it assigns to each point a vector.

If $\vec{F} = \nabla f$ for some function f, we call $\vec{F}$ a gradient vector field and f the potential of the vector field.

E.g. • Gravitational force fields.

$$\vec{F} = -\frac{mMG}{\|\vec{r}\|^3}\vec{r} \quad \text{where } \vec{r}(x,y,z) = (x,y,z).$$

$$f = \frac{mMG}{\|\vec{r}\|}.$$

• $\vec{F} = y\vec{i} + x\vec{j}$ is a gradient vector field with $f(x,y) = xy$.

Q: How to find f?

A: Integrate.

since $\frac{\partial f}{\partial x} = y. \Rightarrow \int \frac{\partial f}{\partial x}dx = \int y\,dx = xy + c$

• $\vec{F} = y\vec{i} - x\vec{j}$ is not a gradient vector field.

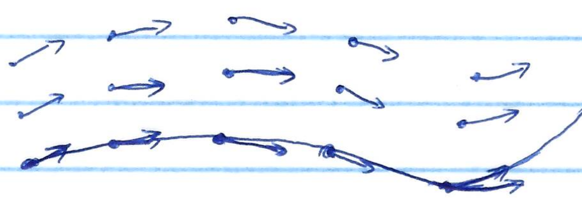
$$\left(\frac{\partial f}{\partial x} = y \text{ and } \frac{\partial f}{\partial y} = -x. \right.$$

$$\Downarrow$$

$$\left. \frac{\partial^2 f}{\partial y \partial x} = 1 \neq \frac{\partial^2 f}{\partial x \partial y} = -1 \right).$$

* **Flow lines:**

Given a vector field $\vec{F}$, a flow line is a path $\vec{c}(t)$ such that $\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$.



E.g. $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$.

Find a flow line.

$$\vec{c}(t) = (x(t), y(t))$$
$$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$
$$-y(t)\vec{i} + x(t)\vec{j} = x'(t)\vec{i} + y'(t)\vec{j}.$$

$$\Rightarrow \begin{cases} -y(t) = x'(t) \\ x(t) = y'(t). \end{cases}$$

$x(t) = \cos(t)$ and $y(t) = \sin(t)$ work.

$$\Rightarrow \vec{c}(t) = (\cos(t), \sin(t))$$

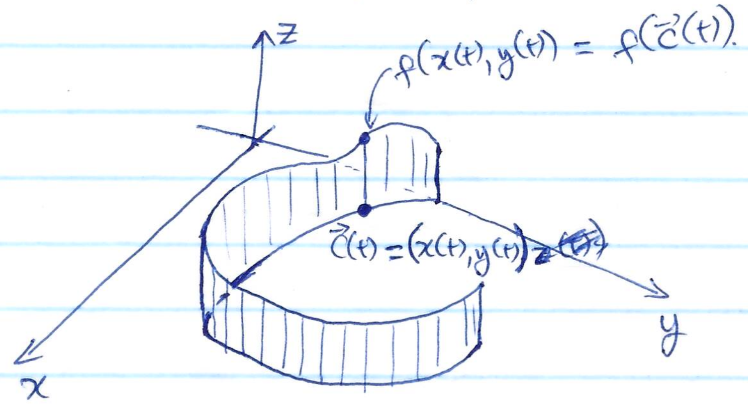
E.g. Let $\vec{F} = x\vec{i} + 2x\vec{j} + 3y\vec{k}$.

Find a flow line.

$$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$
$$(x(t), 2x(t), 3y(t)) = (x'(t), y'(t), z'(t)).$$
$$\Rightarrow x(t) = x'(t) \Rightarrow x(t) = e^t.$$
$$y'(t) = 2x(t) \Rightarrow 2e^t \Rightarrow y(t) = 2e^t.$$
$$z'(t) = 6e^t \Rightarrow z(t) = 6e^t.$$
$$\Rightarrow c(t) = (e^t, 2e^t, 6e^t).$$

Goal of the course: generalize the FTOC to several variables.

Need the concept of a path integral.



"area of the fence" whose base is the image of $\vec{c}$ and height $f(x,y)$ at (x,y).

* 7.1 The path integral.

Def: (2 variables) The path integral, or the integral of $f(x,y)$ along the path $\vec{c}$, is defined by

$$\int_{\vec{c}} f\,ds := \int_a^b f(x(t), y(t))\,\|c'(t)\|\,dt.$$

where $\vec{c}: [a,b] \to \mathbb{R}^2$, $\vec{c}$ is differentiable.

E.g. The base of the fence in the first quadrant

$$\vec{c}: [0, \tfrac{\pi}{2}] \to \mathbb{R}^2$$

$$\vec{c}(t) = (30\cos^3 t,\ 30\sin^3 t).$$

the height of the fence at (x,y) is

$$f(x,y) = 1 + \frac{y}{3}.$$

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Area of fence = $\int_{\vec{c}} \overset{\text{scalar}}{f}\, \underset{\text{arc length}}{ds}$ $\quad ds = \|\vec{c}'(t)\| dt.$

$\boxed{c'(t) = (-90\cos^2 t \sin t, 90\sin^2 t \cos t)}$ $= \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right)\sqrt{90^2\cos^4 t \sin^2 t + 90^2 \sin^4 t \cos^2 t}\, dt$

$$= \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) 90 \cos t \sin t \, dt$$

$$= 90\int_0^{\pi/2} (\sin t + 10\sin^4 t)\cos t \, dt.$$

$u = \sin t$

$du = \cos t \, dt$

$$= 90 \int_0^1 (u + 10u^4)\, du.$$

$$= 90 \left[\frac{u^2}{2} + 2u^5\right]_0^1$$

$$= 90\left(\frac{1}{2} + 2\right)$$

$$= 225.$$

---

3 variables:

$$\int_{\vec{c}} f\, ds := \int_a^b f(x(t), y(t), z(t))\, \|\vec{c}'(t)\|\, dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$.

$f(x, y, z) = x^2 + y^2 + z.$

Evaluate $\int_{\vec{c}} f\, ds.$

Sol: $\int_{\vec{c}} f\, ds = \int_0^{2\pi} f(\vec{c}(t))\, \cancel{dt}\, \|\vec{c}'(t)\|\, dt.$

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$$f(\vec{c}(t)) = f(x(t), y(t), z(t))$$
$$= x^2(t) + y^2(t) + z(t)$$
$$= \cos^2 t + \sin^2 t + t$$
$$= 1 + t.$$

$$\vec{c}'(t) = (-\sin t, \cos t, 1)$$
$$\|\vec{c}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2}$$
$$= \sqrt{2}.$$

$$\Rightarrow \int_{\vec{c}} f\,ds = \int_0^{2\pi} (1+t)\sqrt{2}\,dt = \sqrt{2}\left(t + \frac{t^2}{2}\right)_0^{2\pi}$$
$$= \sqrt{2}\left(2\pi + \frac{(2\pi)^2}{2}\right)$$
$$= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$