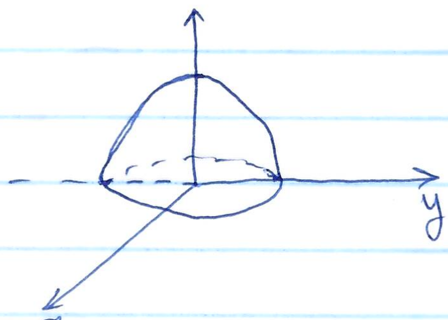
* Double Integrals: (5.1 – 5.4).

$$\iint_R f(x,y)\,\underbrace{dA}_{\substack{dx\,dy \\ \text{or } dy\,dx}} =$$ Volume of the region above R and under the graph of f when f is non-negative.

Example: $f(x,y) = 1 - x^2 - y^2$

$R: -1 \le x \le 1,\ -1 \le y \le 1.$



$$\int_{-1}^{1}\left[\int_{-1}^{1}(1-x^2-y^2)\,dy\right]dx = \int_{-1}^{1}\left(y - x^2y - \frac{y^3}{3}\right)\Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1}\left\{1 - x^2 - \frac{1}{3} - \left(-1 + x^2 + \frac{1}{3}\right)\right\} dx$$

$$= \int_{-1}^{1} 2 - 2x^2 - \frac{2}{3}\,dx$$

$$= \int_{-1}^{1}\left(\frac{4}{3} - 2x^2\right) dx$$

$$= \frac{4}{3}x - \frac{2x^3}{3}\Big|_{-1}^{1}$$

$$= \frac{4}{3} - \frac{2}{3} - \left(-\frac{4}{3} + \frac{2}{3}\right)$$

$$= \frac{8}{3} - \frac{4}{3}$$

$$= \frac{4}{3}$$

↑ iterated integral because it is obtained by integrating with respect to y and then w.r.t. x.

Sometimes evaluating an iterated integral can be hard so we may need to change the order of integration.

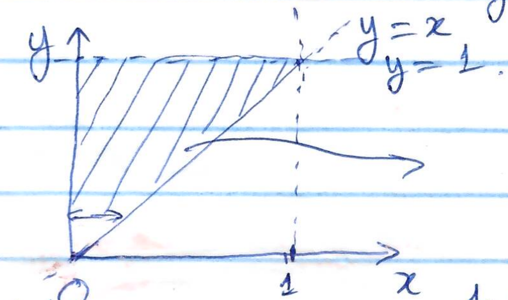
E.g. $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

e^{y^2} doesn't have an antiderivative that we know

$\Rightarrow$ cannot solve directly.

$\Rightarrow$ try changing the order of integration.

$0 \le x \le 1$ and $x \le y \le 1$.



equivalently $0 \le y \le 1$.
$0 \le x \le y$.

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$= \int_0^1 e^{y^2} x \Big|_0^y \,dy$$

$$= \int_0^1 ye^{y^2} - 0 \,dy.$$

$$= \int_0^1 ye^{y^2}\,dy.$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1$$

$$= \frac{1}{2} e^1 - \frac{1}{2} \cdot e^0$$

$$= \frac{1}{2}(e - 1).$$

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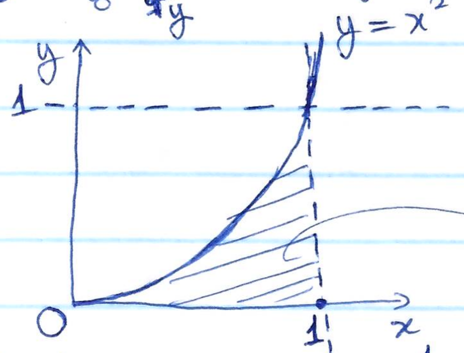
E.g. $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$



$0 \le y \le x^2$
$0 \le x \le 1.$

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^1 e^{x^3} y \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 e^{x^3}(x^2 - 0)\,dx.$$

$$= \int_0^1 x^2 e^{x^3}\,dx$$

or let $u = x^3 \Rightarrow du = 3x^2 dx.$

$$= \frac{e^{x^3}}{3}\Big|_0^1$$

$$= \frac{1}{3}(e - 1).$$

* Triple Integrals: (5.5)

$$\iiint_R f(x,y,z)\, \underbrace{dV}_{dx\,dy\,dz}$$

$R \to$ solid in space

E.g. Let B be the box given by

$0 \leq x \leq 1, \quad 0 \leq y \leq 2, \quad -1 \leq z \leq 0.$

evaluate $\iiint_B x^2 + xy + z^2 y \, dV$

$$= \int_0^1 \int_0^2 \int_{-1}^0 x^2 + xy + z^2 y \; dz\,dy\,dx \qquad \text{(iterated integrals).}$$

$$= \int_0^1 \int_0^2 x^2 z + xyz + \frac{z^3}{3} y \Big|_{z=-1}^{z=0} dy\,dx$$

$$= \int_0^1 \int_0^2 0 - \left(-x^2 - xy - \frac{y}{3}\right) dy\,dx.$$

$$= \int_0^1 \int_0^2 x^2 + xy + \frac{y}{3} \; dy\,dx$$

$$= \int_0^1 x^2 y + \frac{xy^2}{2} + \frac{y^2}{6} \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 2x^2 + \frac{4}{2}x + \frac{4}{6} \; dx.$$

$$= \int_0^1 2x^2 + 2x + \frac{2}{3} \; dx$$

$$= \frac{2x^3}{3} + x^2 + \frac{2}{3}x \Big|_0^1$$

$$= \frac{2}{3} + 1 + \frac{2}{3}$$

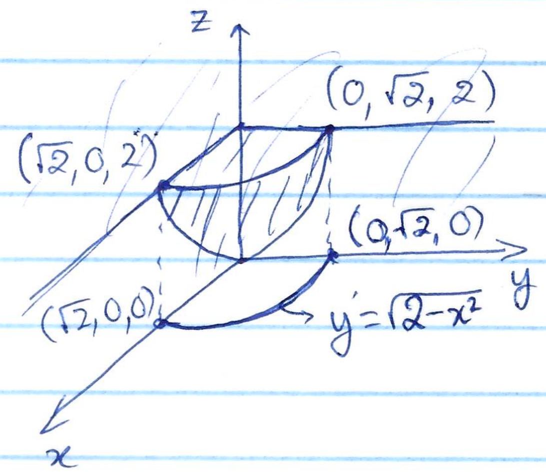
$$= \frac{7}{3}.$$

Exercise: Verify that you get the same ~~number~~ answer if you change the order of integration.

E.g. Let W be the region bounded by the planes $x=0$, $y=0$, $z=2$, and the surface $z=x^2+y^2$. lying in the quadrant $x\geq 0$, $y \geq 0$.

Compute $\iiint_W x\,dx\,dy\,dz$.

$W=\{x^2+y^2\le z\le 2,\ \cancel{\sqrt{2-x^2}}\ 0 \le y\le\sqrt{2-x^2},\ 0\le x\le\sqrt{2}\}$



$$\int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{2} x \; dz\,dy\,dx$$

$$= \int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}} xz \Big|_{z=x^2+y^2}^{z=2} dy\,dx$$

$$= \int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}} 2x - x(x^2+y^2)\; dy\,dx$$

$$= \int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}} 2x - x^3 - xy^2 \; dy\,dx$$

$$= \int_0^{\sqrt2} 2xy - x^3y - x\frac{y^3}{3}\Big|_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_0^{\sqrt2} 2x(\sqrt{2-x^2}) - x^3\sqrt{2-x^2} - x\frac{(\sqrt{2-x^2})^3}{3}\; dx.$$

$$= \int_0^{\sqrt2} 2x(2-x^2)^{1/2} - x^3\sqrt{2-x^2} - \frac{x(2-x^2)\sqrt{2-x^2}}{3}\; dx.$$

$$= \int_0^{\sqrt2} 2x\sqrt{2-x^2} - x^3\sqrt{2-x^2} - \frac{2x\sqrt{2-x^2}}{3} + \frac{x^3\sqrt{2-x^2}}{3}\; dx.$$

$$= \int_0^{\sqrt2} \frac{4x}{3}\sqrt{2-x^2}$$

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$$= \int_0^{\sqrt{2}} (2-x^2)\left(x\sqrt{2-x^2}\right) - \frac{x(2-x^2)^{3/2}}{3}\, dx.$$

$$= \int_0^{\sqrt{2}} \frac{2}{3} x\left(2-x^2\right)^{3/2}\, dx.$$

Let $u = 2-x^2 \Rightarrow du = -2x\,dx.$

$$\hookrightarrow = \int_2^0 \frac{2}{3} u^{3/2} \cdot \frac{du}{-2}$$

$$= -\int_2^0 \frac{u^{3/2}}{3}\, du.$$

$$= \int_0^2 \frac{u^{3/2}}{3}\, du$$

$$= \frac{1}{3} \left.\frac{u^{3/2+1}}{3/2+1}\right|_0^2$$

$$= \frac{1}{3} \frac{2^{5/2}}{5/2}$$

$$= \frac{2}{15} \cdot 2^{5/2}$$

$$= \frac{2^{7/2}}{15}$$

$$= \frac{8\sqrt{2}}{15}.$$