Math 20E: Vector Calculus.

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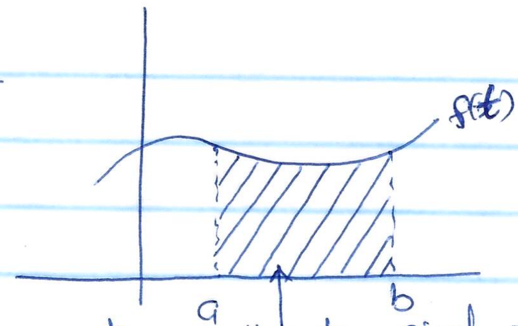
→ this course.

* Contains/will contain: Syllabus
  Exam schedule
  HW
  Office Hours.
  TA Information.
  etc.
* Grading scheme: HW, 2 MTs, Final

  20%, 20%, 20%, 40% Final
  or 20% (HW), 20% (highest MT), 60% Final.
  (whichever is higher)

* Plan: (for the first part of the course).
  - Review:
    Differentiation
    Double Integrals
    Triple Integrals
  - Linear Maps and the change of variable formula.
    ~~Cycli~~ Cylindrical & Spherical Coordinates.
  - Vector fields.
  - Integrals over paths and surfaces.

- Recall:



How to find this area?

$$\text{Area} = \int_a^b f(t)\,dt \overset{FTOC}{=} F(b) - F(a).$$

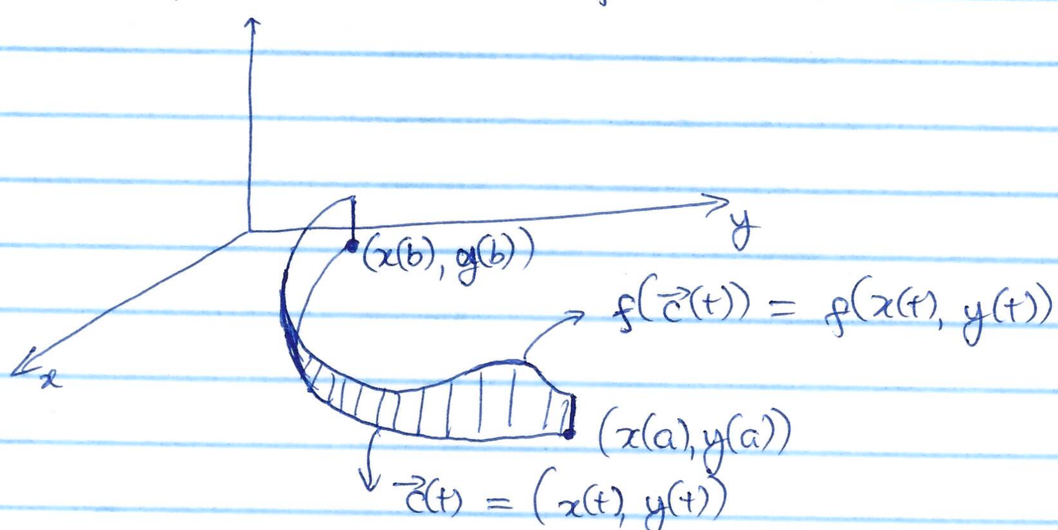
(FTOC)

Fundamental theorem of Calculus: If f is continuous and F is an antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

⇒ ~~In this course, we~~

Area of a fence. Let $f: \mathbb{R}^2 \to \mathbb{R}$.



How to find the area of the fence?

⇒ Fund. Thm of Line Integrals.

$$\int_{\vec{c}} (\nabla f)\cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a)).$$

⇒ In this course, we will build up the mathematical technology to generalize the FTOC.

2.3 Differentiation.

Recall: $f: \mathbb{R} \to \mathbb{R}$

$$\boxed{\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

$\Rightarrow$ Higher dimensions? $\boxed{\text{Partial derivatives}}$

E.g. If $f: \mathbb{R}^2 \to \mathbb{R}$

then $$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

and $$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

~~(What's ∂f~~

E.g. $f(x,y) = xy^3 + \sin(xy)$

Find $\frac{\partial f}{\partial y}(0,1)$.

Sol: $$\frac{\partial f}{\partial y}(x,y) = 3xy^2 + \cos(xy)\cdot x$$

$$\frac{\partial f}{\partial y}(0,1) = 0.$$

Recall: Defn of differentiability (2 variables).

Let $f: \mathbb{R}^2 \to \mathbb{R}$. We say that f is differentiable at (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist and if

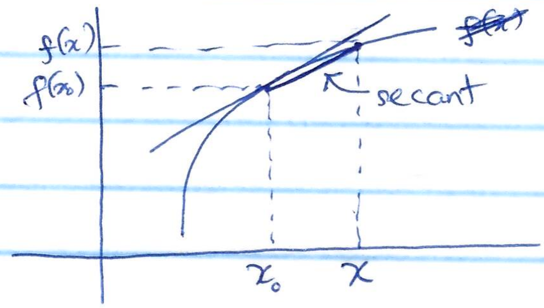
$$\frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y)-(x_0,y_0)\|} \to 0$$

as $(x,y) \to (x_0,y_0)$

What does this mean?

* 1-variable $\quad \dfrac{f(x) - f(x_0) - \left[\frac{df}{dx}(x_0)\right](x - x_0)}{x - x_0} \to 0$ as $x \to x_0$

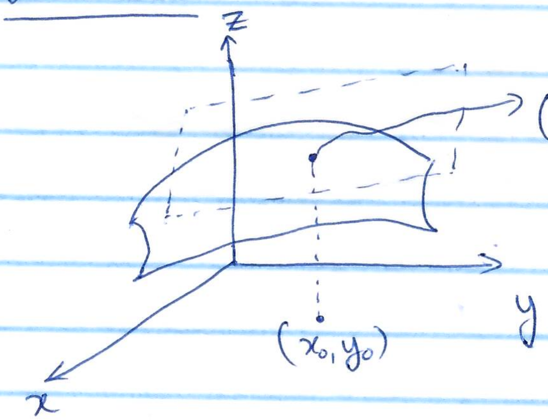
$\Rightarrow \quad \underbrace{\dfrac{f(x) - f(x_0)}{x - x_0}}_{\text{slope of secant}} \to \underbrace{\dfrac{df}{dx}(x_0)}_{\text{slope of tangent}}$ as $x \to x_0$.



$f(x) \to \overbrace{f(x_0) + \left[\frac{df}{dx}(x_0)\right](x - x_0)}^{\text{lin. approx at } x_0}$.

as $x \to x_0$.

* 2-variables:



when we're close to (x_0, y_0), the graph of the tangent plane is close to the graph of f.

we have $\quad (*) \quad z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$

the linear approx of f at (x_0, y_0) (plane) is a good approx. of $f(x)$ when $(x, y) \to (x_0, y_0)$.

Def: (tangent plane) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0). The plane in $\mathbb{R}^3$ given by $(*)$ is called the tangent plane to the graph of f at $(x_0, y_0, f(x_0, y_0))$

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E.g. Find the plane tangent to the graph of
$$f(x,y) = x + y^2 + \cos(xy) \quad \text{at } (0,1).$$

Sol: The equation of the tangent plane (at $(0,1)$) is given by

$$z = \underset{?}{f(0,1)} + \underset{?}{\left[\frac{\partial f}{\partial x}(0,1)\right]}(x-0) + \underset{?}{\left[\frac{\partial f}{\partial y}(0,1)\right]}(y-1)$$

$$\cancel{\frac{\partial f}{\partial x}(x} \quad f(0,1) = 0 + 1^2 + \cos(0)$$
$$= 1 + 1$$
$$= 2$$

$$\frac{\partial f}{\partial x}(x,y) = 1 + 0\cancel{2y^2} + - \cancel{y}\, y\sin(xy)$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,1) = 1 \cancel{+} \cancel{+} - \sin(0) = \cancel{0}\, 1.$$

$$\frac{\partial f}{\partial y}(x,y) = 0 + 2y - x\sin(xy)$$

$$\Rightarrow \frac{\partial f}{\partial y}(0,1) = 2.$$

Hence,
$$z = 2 + (x-0) + 2(y-1).$$
$$= x + 2y.$$

* Differentiability: The general case

The derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$ at $\vec{x}_0$ denoted by $Df(\vec{x}_0)$ is a <u>matrix</u> T whose elements are $t_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\vec{x}_0}$.

i.e. if $f = (f_1, \dots, f_m)$

$$T = Df(\vec{x}_0) = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\vec{x}_0} \\ \vdots & & & \\ \left.\frac{\partial f_m}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_m}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_{\vec{x}_0} \end{bmatrix}$$

derivative or differential of f at $\vec{x}_0$.

<u>Def</u>: Let U be an open set in $\mathbb{R}^n$ and $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Then f is <u>differentiable</u> at $\vec{x}_0 \in U$ if the partial derivatives of f exist at $\vec{x}_0$ and if

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0$$

E.g. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (\underbrace{x + e^z + \sin y}_{f_1}, \underbrace{yx^2}_{f_2})$

Find $Df(x, y, z)$.

$$Df(x, y, z) = \begin{bmatrix} \frac{\partial (x + e^z + \sin y)}{\partial x} & \frac{\partial (x + e^z + \sin y)}{\partial y} & \frac{\partial (x + e^z + \sin y)}{\partial z} \\ \frac{\partial (yx^2)}{\partial x} & \frac{\partial (yx^2)}{\partial y} & \frac{\partial (yx^2)}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos y & e^z \\ 2yx & x^2 & 0 \end{bmatrix}$$

7.

Remark: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ then $Df(x_0)$ is a $1 \times n$ matrix. The corresponding vector $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is called ~~to~~ the gradient and denoted by ∇f.

Thm: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in U$, then f is continuous at $\vec{x}_0$.

Thm: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is such that

- the partials $\frac{\partial f_i}{\partial x_j}$ all exist
- and are continuous in a neighborhood of $\vec{x} \in U$.

then f is differentiable at $\vec{x}$.

$\hookrightarrow$ much easier to check than the def. of diff.

E.g. $f(x, y, z) = (x + e^z + \sin y,\ y x^2)$ is differentiable because the partial derivatives exist and ~~con~~ are continuous.

* 2.5. Properties of the derivative

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x}_0$. Then:

(1) Constant multiple rule:

If $c \in \mathbb{R}$ and $h(\vec{x}) = c f(\vec{x})$, then h is differentiable at $\vec{x}_0$ and $Dh(\vec{x}_0) = c Df(\vec{x}_0)$.

(2) Sum rule: If $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$, then h is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = Df(\vec{x}_0) + Dg(\vec{x}_0)$$

(3) Product rule: ~~If $h(\vec{x}) = f(\vec{x})\, g$~~

If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{x}_0$ and $h(\vec{x}) = f(\vec{x})\, g(\vec{x})$, then h is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = \underbrace{g(\vec{x}_0)}_{\in \mathbb{R}} \underbrace{Df(\vec{x}_0)}_{1\times n \text{ matrix}} + \underbrace{f(\vec{x}_0)}_{\in \mathbb{R}} \underbrace{Dg(\vec{x}_0)}_{1\times n \text{ matrix}}.$$

~~Eg.~~

(4) Quotient Rule:

(4) With the same assumption as in rule (3), suppose further that $g(\vec{x}) \neq 0 \quad \forall \vec{x} \in U$

if $h(\vec{x}) = \dfrac{f(\vec{x})}{g(\vec{x})}$, then h is differentiable at $\vec{x}_0$

and

$$Dh(\vec{x}_0) = \frac{g(\vec{x}_0)\, Df(\vec{x}_0) - f(\vec{x}_0)\, Dg(\vec{x}_0)}{[g(\vec{x}_0)]^2}.$$

E.g. $f(x,y) = x^2 + y^2$ and $g(x,y) = \sin^2 x + 1$

$$h(x,y) = \frac{x^2+y^2}{\sin^2 x + 1}$$

Find $Dh(x,y)$

Sol:
$$Dh(x,y) = \frac{g(x,y)\,Df(x,y) - f(x,y)\,Dg(x,y)}{[g(x,y)]^2}$$

$$\cancel{= (\sin^2 x+1)[2x, 2y] - (x^2+y^2)[}$$

$$= \frac{(\sin^2 x + 1)\left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] - (x^2+y^2)\left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right]}{[g(x,y)]^2}$$

$$= \frac{(\sin^2 x + 1)[2x \quad 2y] - (x^2+y^2)[2\sin x\cos x \quad 0]}{(\sin^2 x + 1)^2}$$

$$= \left[\frac{(\sin^2 x+1)\cdot 2x - (x^2+y^2)\,2\sin x\cos x}{(\sin^2 x + 1)^2} \qquad \frac{\cancel{(\sin^2 x+1)}\cdot 2y}{(\sin^2 x+1)^{\cancel{2}}}\right].$$

(5) Chain rule:

Let $f: \underset{\text{open set}}{V} \subset \mathbb{R}^m \to \mathbb{R}^p$ and $g: \underset{\text{open set}}{U} \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$

Let g be diff. at $\vec{x}_0$ and f be diff. at $\vec{y}_0 = g(\vec{x}_0)$.

Then

$$\underbrace{D(f\circ g)(\vec{x}_0)}_{p\times n \text{ matrix}} = [\underbrace{Df(\vec{y}_0)}_{\substack{p\times m\\ \text{matrix}}}][\underbrace{Dg(\vec{x}_0)}_{\substack{m\times n\\ \text{matrix}}}].$$

Compare to single variable calculus $\dfrac{df(g(x_0))}{dx} = f'(\underbrace{g(x)}_{y})\,g'(x)$.

E.g. $g(x,y) = (x^2+1,\ y^2)$ and $f(u,v) = (u+v,\ u,\ v^2)$

Find $D(f\circ g)(1,1)$ using the chain rule.

Soln:

$\cancel{D(f\circ g)(x,y)}$ By the chain rule

$$D(f\circ g)(1,1) = [Df(g(1,1))][Dg(1,1)]$$

$$= \overset{?}{[Df(2,1)]}\overset{?}{[Dg(1,1)]} \quad \text{since } g(1,1)=(2,1)$$

$$Df(u,v) = \begin{bmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u+v)}{\partial v} \\ \frac{\partial(u)}{\partial u} & \frac{\partial(u)}{\partial v} \\ \frac{\partial(v^2)}{\partial u} & \frac{\partial(v^2)}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \Rightarrow \cancel{Df(2,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}}$$

$$Df(2,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \Leftarrow$$

and $$Dg(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \Rightarrow Dg(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D(f\circ g)(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

* Special case of the chain rule:

$c: \mathbb{R} \to \mathbb{R}^3$ is a diff. path, $f: \mathbb{R}^3 \to \mathbb{R}$

$\vec{c}(t) = (x(t), y(t), z(t))$

and $h(t) = f(c(t)) = f(x(t), y(t), z(t))$.

then

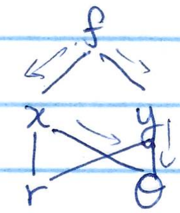
$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \nabla f(\vec{c}(t)) \cdot \vec{c'(t)}$$

$$= \underbrace{[Df(\vec{c}(t))]}_{1\times 3}\underbrace{[D\vec{c}(t)]}_{3\times 1}.$$

E.g. Let $f(x,y)$ be given and make the substitution $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial r}$.

Sol:



$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \left(\frac{\partial f}{\partial x}\right)\cdot(-r\sin\theta) + \left(\frac{\partial f}{\partial y}\right)(r\cos\theta)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = (\cos\theta)\,\frac{\partial f}{\partial x} + (\sin\theta)\,\frac{\partial f}{\partial y}.$$