Vector Calculus 20E, Fall 2014, Lecture A, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let D be the upper half of the unit disc, given by $x^2 + y^2 \leq 1, y \geq 0$. Find the average of the function $f(x, y) = y$ over D.

2. Let D^* be the right-hand half of the unit disc, given by $x^2 + y^2 \leq 1, x \geq 0$. Let $D = T(D^*)$, where T is the map $(u, v) \mapsto (u^2 - v^2, 2uv)$. Calculate the area of D.

3. Let C be the curve in the plane described by $t \mapsto (\cos^3 t, \sin t)$ for $0 \leq t \leq 2\pi$. Use Green's theorem to compute the area enclosed by C.

4. Let Σ be the part of the cone $z = \sqrt{x^2 + y^2}$ lying above the standard unit square $0 \leq x, y \leq 1$. Compute the surface area of Σ.

5. Let C be the oriented triangular path formed by travelling from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$ and then back to $(1, 0, 0)$ along straight line segments. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y, x, x^2)$. Compute the circulation of $\mathbf{F}$ around C:

$$\int_C \mathbf{F}.\mathrm{d}\mathbf{s}$$

6. Let γ be the oriented path $t \mapsto (\sqrt{1 + t^2}, \sqrt[3]{1 + t^3}, \sqrt[4]{1 + t^4})$ for $0 \leq t \leq 1$. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (yz, xz, xy)$. Is $\mathbf{F}$ conservative? Calculate

$$\int_\gamma \mathbf{F}.\mathrm{d}\mathbf{s}$$

7. Let Σ be the part of the unit sphere $x^2 + y^2 + z^2 = 1$ with $x, y, z \geq 0$, oriented outwards from the origin as usual. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y, -x, 1)$. Compute the flux of $\mathbf{F}$ out of Σ:

$$\int_\Sigma \mathbf{F}.\mathrm{d}\mathbf{S}$$

8. Let Σ be the surface made by gluing the upper unit hemisphere (given by $x^2+y^2+z^2 = 1, z \geq 0$) onto the unit disc in the xy-plane (given by $x^2 + y^2 \leq 1, z = 0$); orient the whole surface outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x^2, xz, 3z)$. Compute the flux of $\mathbf{F}$ out of Σ:

$$\int_\Sigma \mathbf{F}.\mathrm{d}\mathbf{S}$$