Math 20E - Lecture A00 **Final Exam, VERSION A**
Fall 2016 **Name:** ________________
12/06/2016 PID: ________________
Time Limit: 3 Hours Section Time: ________________

This exam contains 5 pages (including this cover page) and 10 questions.
Total of points is 100.

You may not use any notes (except your cheat sheet) or calculators during this exam.
Write your *Name, PID, and Section* on the front of your Blue Book.
Write the *Version* of your exam on the front of your Blue Book.
Write your solutions clearly in your Blue Book.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.

1. (3 points) Carefully read and complete the instructions at the top of this exam sheet.
   Just give all 3 points.

2. (10 points) Using Green's theorem, or otherwise, to evaluate $\int_C x^3\,dy - y^3\,dx$, where C is the unit circle $x^2+y^2=1$ with the counterclockwise orientation.
   Solution. Note that $Q(x,y)=x^3$ and $P(x,y)=-y^3$. By Green's theorem,

$$\begin{aligned}\int_{C^+} x^3dy - y^3dx &= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \\ &= \int\int_D 3x^2+3y^2\,dxdy \\ &= 3\int_0^{2\pi}\int_0^1 r^2r\,drd\theta \\ &= \frac{3}{4}2\pi \\ &= \frac{3\pi}{2}.\end{aligned}$$

   - 3 points: for stating Q and P either explicitly or implicitly.
   - 4 points: for using Green's theorem.
   - 3 points: for calculating the integral.

3. (20 points) Let $\mathbf{F}(x,y,z) = (3x^2+2xy)\mathbf{i} + (2yz^2+x^2+3y)\mathbf{j} + (2y^2z)\mathbf{k}$.

   a. (5 points) Compute $\nabla\cdot\mathbf{F}$.
   Solution.

$$\begin{aligned}&\text{(3 points for stating the formula)} \quad \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(3x^2+2xy) + \frac{\partial}{\partial y}(2yz^2+x^2+3y) + \frac{\partial}{\partial z}(2y^2z) \\ &\text{(2 points for calculation)} \quad = 6x+2y+2z^2+3+2y^2.\end{aligned}$$

b. (5 points) Compute $\nabla \times \mathbf{F}$.
Solution. (3 points for stating the formula and 2 points for calculation).

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+2xy & 2yz^2+x^2+3y & 2y^2z \end{vmatrix} \\ &= (4yz-4yz)\mathbf{i} - (0-0)\mathbf{j} + (2x-2x)\mathbf{k} \\ &= 0.\end{aligned}$$

c. (5 points) Is $\mathbf{F}$ conservative? If yes, find the function f such that $\mathbf{F} = \nabla f$.
Solution. (It is fine if they only give an explicit function f without calculating it. But if they solve it step by step and get a wrong answer, give them partial credits.) Since $\nabla \times \mathbf{F} = 0$, $\mathbf{F}$ is conservative.

$$f(x,y,z) = x^3 + x^2y + y^2z^2 + \frac{3}{2}y^2.$$

d. (5 points) Let $\mathbf{c}(t) = (\cos t, \sin t, 0)$ for $0 \le t \le \pi$. Compute the line integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.
Solution. (3 points for stating the formula and 2 points for calculation) By the fundamental theorem of line integral,

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} \\ &= f(\cos\pi, \sin\pi, 0) - f(\cos 0, \sin 0, 0) \\ &= f(-1,0,0) - f(1,0,0) \\ &= -2.\end{aligned}$$

4. (10 points) Let S be the hemisphere $x^2+y^2+z^2 = 9, z \ge 0$, oriented with the upward normal. Let $\mathbf{F}$ be the vector field $(x^2+z)\mathbf{i} + 3z(e^{x^2+y^2} + x^{10})\mathbf{j} + 2y^{15}z\mathbf{k}$. Compute the integral $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.
Solution. (2 points) The boundary curve of S is the circle $x^2+y^2 = 9$ which can be parametrized by $\mathbf{c}(\theta) = (3\cos\theta, 3\sin\theta, 0)$ for $0 \le \theta \le 2\pi$. By Stoke's theorem,

$$\begin{aligned}\text{(3 points)} \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \\ \text{(2 points)} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \int_0^{2\pi} (9\cos^2\theta, 0, 0) \cdot (-3\sin\theta, 3\cos\theta, 0) d\theta \\ &= -27 \int_0^{2\pi} \cos^2\theta \sin\theta d\theta \\ \text{(3 points points for calculation)} &= 0.\end{aligned}$$

5. (10 points) Find the flux of the vector field $\mathbf{F}(x,y,z) = x^3y\mathbf{i} + z^8\mathbf{j} - 3x^2yz\mathbf{k}$ out of the surface of the standard unit cube $(0 \le x,y,z \le 1)$ in $\mathbb{R}^3$. (Hint: use Gauss' Theorem)
Solution. Let W be a unit solid ball, so S is the boundary surface of W. By Gauss' theorem,

the flux is

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int\int\int_W \operatorname{div}(\mathbf{F})\,dV\\ &= \int\int\int_W (3x^2y + 0 - 3x^2y)\,dV\\ &= 0.\end{aligned}$$

4 points for stating/writing Gauss' theorem.
4 points for calculating $\operatorname{div}(\mathbf{F})$.
2 points for calculating the final integral.

6. (10 points) Let $\mathbf{c}$ be the path given by $\mathbf{c}(t) = (t, \cos t, \sin t)$ for $0 \le t \le \frac{\pi}{2}$. Find the length of the path.
Solution. The length of the path $\mathbf{c}$ is

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \|\mathbf{c}'(t)\| dt &= \int_0^{\frac{\pi}{2}} \sqrt{1+\sin^2 t + \cos^2 t} dt\\ &= \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

4 points for the formula of the length.
3 points for finding $\mathbf{c}'(t)$ (either implicitly or explicitly).
3 points for calculating.

7. (10 points) Evaluate the line integral $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}(x,y,z) = (-y, x, e^{-z})$ and $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$.
Solution.

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t))\cdot \mathbf{c}'(t) dt\\ &= \int_0^{2\pi} (-\sin t, \cos t, e^{-t})\cdot(-\sin t, \cos t, 1) dt\\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t + e^{-t} dt\\ &= \int_0^{2\pi} (1+e^{-t}) dt\\ &= 2\pi - e^{-2\pi} + 1.\end{aligned}$$

3 points for finding $\mathbf{F}(\mathbf{c}(t))$.
3 points for finding $\mathbf{c}'(t)$.
4 points for calculation.

8. (10 points) Let S be the surface $z = x^2 + y^2, 0 \le z \le 4$. Evaluate the following integral $\int\int_S \frac{y}{\sqrt{z}}\, dS$.
Solution. The parametrization of S is $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{T}_r = (\cos\theta, \sin\theta, 2r)$, $\mathbf{T}_\theta = (-r\sin\theta, r\cos\theta, 0)$, $\mathbf{T}_r \times \mathbf{T}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$,

and $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = r\sqrt{1+4r^2}$. Hence,

$$\begin{aligned}\int\int_S \frac{y}{\sqrt{z}}\,dS &= \int_0^{2\pi}\int_0^2 \frac{r\sin\theta}{\sqrt{r^2}} r\sqrt{1+4r^2}\,dr d\theta \\ &= \int_0^{2\pi}\sin\theta\,d\theta \int_0^2 r\sqrt{1+4r^2}\,dr \\ &= 0.\end{aligned}$$

3 points for $\Phi(r,\theta)$ (or other reasonable parametrization).
3 points for $\|T_r \times T_\theta\|$
2 points for expressing the integral in term of r and θ
2 points for calculation.

9. (10 points) Let S be the part of the cone $z = \sqrt{x^2+y^2}$ lying above the standard unit square $0 \le x, y \le 1$. Compute the surface area of S.
Solution. Parametrization $\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$. Then $\|T_x \times T_y\| = \sqrt{2}$. Thus, the surface area of S is

$$\int\int_S \|T_x \times T_y\| dS = \int_0^1\int_0^1 \sqrt{2}\,dxdy = \sqrt{2}.$$

3 points for $\Phi(x,y)$ (or other reasonable parametrization).
3 points for $\|T_x \times T_y\|$
2 points for expressing the integral in term of x and y
2 points for calculation.
For this problem, I expect that many would parametrize the surface by using polar coordinates which might lead to wrong limits for the integral. If they do that, you can take 2 points off.

10. (7 points) Let S be a surface in $\mathbb{R}^3$, and let ∂S be the boundary of S. Let $\mathbf{F}$ be a vector field on S with continuous partial derivatives. Suppose that you are given the following information about S and $\mathbf{F}$:

i. S lies in the plane $y = 3$.
ii. Area$(S) = 17$.
iii. Length$(\partial S) = 25$.
iv. $\operatorname{div}(\mathbf{F}) = x^2 + y^2 - z$.
v. $\operatorname{curl}(\mathbf{F}) = 3x\mathbf{i} - y\mathbf{j} - 2z\mathbf{k}$.

Using this information, evaluate the absolute value of the line integral $\int_{\partial S} \mathbf{F}\cdot d\mathbf{s}$.
Solution. By Stoke's theorem,

$$\begin{aligned}\text{(2 points)} \int_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int\int_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} \\ \text{(2 points)} &= \int\int_S (3x, -y, 2z)\cdot(0,1,0)dS \\ &= \int\int_S -y\,dS \\ \text{(2 points)} &= -3\int\int_S dS \\ &= -3\text{Area}(S) \\ \text{(1 point)} &= -51.\end{aligned}$$

Both answers -51 and 51 are fine.