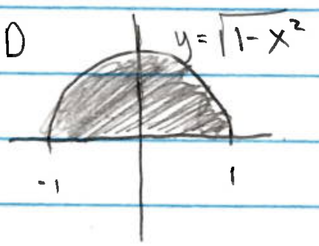
Fall 2014

EXERCISE 1

Recall that if f is integrable over the plane region D, then the average value of f over R is $\frac{1}{A(D)} \iint_D f(x,y)\,dA$

D



$$A(D) = \frac{\text{area of circle with } r=1}{2}$$

$$= \frac{\pi(1)^2}{2} = \frac{\pi}{2}$$

$$\iint_D f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y \; dy\,dx$$

$$= \int_{-1}^{1} \frac{1}{2}y^2 \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left[(1-x^2) - 0\right] dx$$

$$= \frac{1}{2}\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

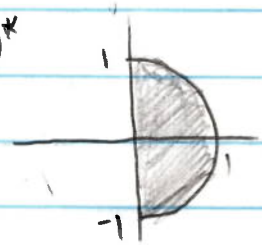
$$= \frac{1}{2}\left[2 - \frac{2}{3}\right] = 1 - \frac{1}{3} = \frac{2}{3}.$$

Then average value $= \underline{\frac{2}{3}}$

JJCG

EXERCISE 2

D^*



$$D = T(D^*)$$

with $T(u,v) \to (u^2 - v^2, 2uv)$

Now $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

Thus,

$$\iint_D dx\,dy = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_{-1}^{1} \int_0^{\sqrt{1-v^2}} |4u^2 + 4v^2| \, du\,dv$$

$u = r\cos\theta$

$y = r\sin\theta$

$$\left| \frac{\partial(u,v)}{\partial(r,\theta)} \right| = r$$

$$= \int_{-1}^{1} \int_0^{\sqrt{1-v^2}} 4(u^2 + v^2)\, du\,dv$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 4r^2 \cdot r \, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \cdot \int_0^1 4r^3 dr = \pi \cdot r^4 \Big|_0^1 = \pi$$

EXERCISE 3

$x = \cos^3 t \qquad 0 \le t \le 2\pi$

$y = \sin t$

by Thm 2 in Sec 8.1

$$A = \frac{1}{2}\int_0^{2\pi}\left[\cos^3 t(\cos t) - \sin t\left(3\cos^2 t\cdot(-\sin t)\right)\right] dt$$

$$= \frac{1}{2}\int_0^{2\pi} \cos^4 t + 3\cos^2 t \sin^2 t \; dt$$

$$= \frac{1}{2}\int_0^{2\pi} \cos^4 t + 3\cos^2 t\,(1-\cos^2 t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 3\cos^2 t - 2\cos^4 t \; dt$$

$$= \frac{1}{2}\left[\int_0^{2\pi} 3\left(\frac{1+\cos 2t}{2}\right) dt - 2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)^2 dt\right.$$

$$= \frac{3}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos 2t + \cos^2 2t\right) dt$$

$$= \frac{3}{2}(2\pi) - \frac{1}{2}(t + \sin 2t)\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 4t}{2}\, dt$$

$$= 3\pi - \pi \quad - \frac{1}{4}\left(t + \frac{1}{4}\sin 4t\right)\Big|_0^{2\pi}$$

$$= 2\pi - \frac{1}{4}(2\pi) = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$

EXERCISE 4

Since the surface S is a graph in the form $z = g(x,y)$ where $(x,y) \in D$ (D being the unit square)

$$A(S) = \iint_D \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right) + 1} \right) dA$$

$$= \int_0^1 \int_0^1 \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1} \; dx\,dy.$$

$$\int_0^1 \int_0^1 \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} \; dx\,dy$$

We're okay since $(0,0)$ is on the boundary.

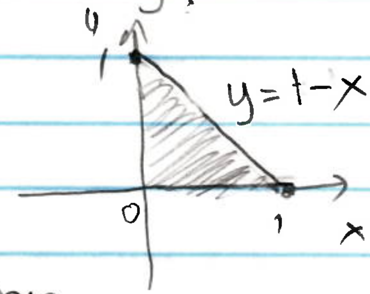
$$= \int_0^1 \int_0^1 \sqrt{2} \; dx\,dy = \sqrt{2}$$

EXERCISE 5

We could parametrize the lines and perform the computation directly but its easy to see that the plane that contains our surface is given by

$x + y + z = 1 \Rightarrow z = 1 - x - y.$

and D is described by



so we apply Stokes" theorem

$$\int_C \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & x^2 \end{vmatrix} = (0, -2x, 0)$$

$\phi(u,v) = (u, v, 1-u-v).$

$0 \leq u \leq 1$

$0 \leq v \leq 1-u$

$T_u = (1, 0, -1)$

$T_v = (0, 1, -1)$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$ has correct orientation.

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^1 \int_0^{1-u} (0, -2u, 0) \cdot (1,1,1)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} -2u\, dv\, du = \int_0^1 -2u(1-u)\, du = \int_0^1 -2u + 2u^2\, du$$

$$= -u^2 + \frac{2}{3}u^3 \Big|_0^1 = -\frac{1}{3}$$

EXERCISE 6.

Following the hint to check if $\vec{F}$ is conservative we can see

$\vec{F} = \nabla f$ for $f = xyz$

(we could also check if $\nabla \times F = 0$).

In any case $\vec{F}$ is conservative, so we don't need to "use" the oriented path given.

So we have

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\gamma \nabla f \cdot d\vec{s} = f(\gamma(b)) - f(\gamma(a))$$

$\gamma(a) = (1,1,1)$

$\gamma(b) = (\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2})$

$a = 0 \quad b = 1$

$$= 2^{1/2} 2^{1/3} 2^{1/4} - (1)(1)(1)$$

$$= 2^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} - 1$$

$$= 2^{\frac{6+4+3}{12}} - 1$$

$$= 2^{13/12} - 1$$

EXERCISE 7

We parametrize the surface by

$$\Phi = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

with $0 \leq \theta \leq \pi/2 \qquad 0 \leq \phi \leq \pi/2$

since $x \geq 0, y \geq 0, z \geq 0.$

We want to compute $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, \|T_\phi \times T_\theta\| d\theta d\phi$

$\vec{n} = (x, y, z)$ outer normal, $\|T_\phi \times T_\theta\| = 1 \sin\phi$

$F \cdot \vec{n} = (y, -x, 1) \cdot (x, y, z) = z$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi/2} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \sin^2\phi \Big|_0^{\pi/2}$$

$$= \frac{\pi}{4} \left((\sin(\pi/2))^2 - (\sin(0))^2 \right)$$

$$= \pi/4 (1 - 0) = \pi/4$$

EXERCISE 8. We could follow the same process as problem 4 in Midterm 2 (see solution online). However, we now have at our disposal Gauss' Theorem

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot F)\, dV.$$

We have the upper half of the unit sphere with it base. We know $0 \le z \le \sqrt{1-x^2-y^2}$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ $\quad -1 \le x \le 1$

$$\nabla\cdot F = \underbrace{2x + 0 + 3}$$

$$\iint_{\Sigma} \vec{F}\cdot ds = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} 2x \; dz\,dy\,dx + \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} 3 \; dz\,dy\,dx$$

$$= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\sqrt{1-x^2-y^2}\, dy\,dx + 3\int_{0}^{2\pi}\int_{0}^{1} r(1-r^2)^{1/2}\, dr\, d\theta$$

$\left[\begin{array}{l} u = 1-r^2 \\ du = -2r\,dr \end{array}\right]$

$$= 2\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x(1-x^2-y^2)^{1/2}\, dx\, dy - \frac{3}{2}\int_{0}^{2\pi}\int_{1}^{0} u^{1/2}\, du\, d\theta$$

$$= 2\int_{-1}^{1} -\frac{1}{2}\cdot\frac{2}{3}(1-x^2-y^2)^{3/2}\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy - \frac{3}{2}\cdot\frac{2}{3}\cdot 2\pi\, u^{3/2}\Big|_{1}^{0}$$

$$= -2\int_{-1}^{1}\left(1-(1-y^2)-y^2\right)^{3/2} - \left(1-(1-y^2)-y^2\right)^{3/2} + 2\pi$$

$$= 0 + 2\pi = +2\pi$$