Spring 2013, Final exam

Answers/Solutions not guaranteed!!

1) Let γ be closed curve given by $x=t^2-t$, $y=2t^3-3t^2+t$ $0\le t\le 1$.
Use green's to find area of enclosed curve

Stokes/ Greens: $\int_\gamma \bar{F}\cdot d\bar{s} = \iint_S (\nabla\times F)\cdot d\bar{S}$ where $\partial S=\gamma$

Let $F= -y\,dx + x\,dy$

Then $\nabla\times F = \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\hat{z} = 2\hat{z}$

So $\int_\gamma \bar{F}\cdot d\bar{s} = 2A(S)$ if S is area bounded by γ in plane.

Thus $A(S) = \frac{1}{2}\int_\gamma \bar{F}\cdot d\bar{s}$

$$= \frac{1}{2}\int_0^1 dt\left(F(c(t))\cdot c'(t)\right)$$

$$= \frac{1}{2}\int_0^1 dt\ \left(-(2t^3-3t^2+t),\ t^2-t\right)\cdot\left(2t-1,\ 6t^2-6t+1\right)$$

$$= \frac{1}{2}\int_0^1 dt\left(-(2t^3-3t^2+t)(2t-1) + (t^2-t)(6t^2-6t+1)\right)$$

$$= \frac{1}{2}\int_0^1 dt\left(-4t^4+2t^3+6t^3-3t^2-2t^2+t+6t^4-6t^3+t^2\right.$$
$$\left.-6t^3+6t^2-t\right)$$

$$= \frac{1}{2}\int_0^1 dt\left(2t^4-4t^3+2t^2\right) = \frac{1}{2}\left(\frac{2t^5}{5}-\frac{4t^4}{4}+\frac{2t^3}{3}\Big|_0^1\right)$$

$$= \frac{1}{2}\left(\frac{2}{5} - 1 + \frac{2}{3}\right) = \frac{1}{2}\left(\frac{6}{15} - \frac{15}{15} + \frac{10}{15}\right) = \boxed{\frac{1}{30}}$$

2) Find $\int_\gamma \bar{F} \cdot d\bar{s}$, $\bar{F} = y\hat{i} + x\hat{j} + z\hat{k}$,

$\gamma = (2\cos t, 2\sin t, t) \quad t \in [0, 2\pi]$

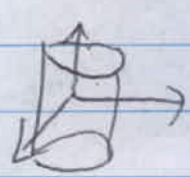
$$\int_\gamma \bar{F} \cdot d\bar{s} = \int_0^{2\pi} dt \; F(c(t)) \cdot c'(t) = \int_0^{2\pi} dt \left(2\sin t, 2\cos t, t\right) \cdot \left(-2\sin t, 2\cos t, 1\right)$$

$$= \int_0^{2\pi} \left(-4\sin^2(t) + 4\cos^2(t) + t\right) dt = \int_0^{2\pi} \left(-4\left(\frac{1-\cos(2t)}{2}\right) + 4\left(\frac{1+\cos(2t)}{2}\right) + t\right)$$

$$= \int_0^{2\pi} dt \left(-2 + 2 + 4\cos(2t) + t\right) = \int_0^{2\pi} dt \left(t + 4\cos(2t)\right)$$

$$= \frac{t^2}{2} + 2\sin(2t) \Big|_0^{2\pi} = \boxed{\frac{(2\pi)^2}{2}}$$

3) Find $\int_\Sigma y^2 \, dA$, Σ part of cylinder $x^2+y^2=4$ b/w $z=0$ and $z=x+3$



Let $\Phi(\theta, z) = (2\cos(\theta), 2\sin(\theta), z)$ for

$0 \le \theta \le 2\pi$, $\quad 0 \le z \le 2\cos\theta + 3$

Then $\int_\Sigma y^2 dA = \int_0^{2\pi} d\theta \int_0^{2\cos\theta+3} dz \; (2\sin(\theta))^2 \, \| T_\theta \times T_z \| \cdot$

$$\|T_\theta \times T_z\| = \left\| \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \right\| = \|2\cos\theta\,\hat{x} + 2\sin\theta\,\hat{y}\|$$

$$= \left(4\cos^2 + 4\sin^2\right)^{1/2} = 2$$

Thus $I = \int_0^{2\pi} d\theta \int_0^{2\cos\theta + 3} dz \left(2^2 \sin^2\theta\right) 2$

$$= \int_0^{2\pi} d\theta \quad 8\sin^2\theta \; z \Big|_0^{2\cos\theta+3}$$

$$= \int_0^{2\pi} d\theta \quad 8\sin^2\theta \left(2\cos\theta + 3\right)$$

$$= \int_0^{2\pi} d\theta \left(16\sin^2\theta\cos\theta + 24\sin^2\theta\right) = \int_0^{2\pi} d\theta \left(16\sin^2\theta\cos\theta + 24\left(\frac{1-\cos(2\theta)}{2}\right)\right)$$

$$= \frac{16}{3}\sin^3\theta + 12\theta - 6\sin(2\theta)\Big|_0^{2\pi}$$

$$= 12(2\pi) = \boxed{24\pi}$$

4) Let D be unit disc in xy plane, Σ be part of graph of $z = xy$ over D. Find surface area of Σ

Let $\Phi(x,y) = (x, y, xy)$ for $\{x^2 + y^2 \le 1\} = D$

Then $A(\Sigma) = \iint_D \|T_x \times T_y\| \, dA$

$$\|T_x \times T_y\| = \left\| \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} \right\| = \| -y\hat{x} - x\hat{y} + \hat{z} \|$$
$$= (y^2 + x^2 + 1)^{1/2}$$

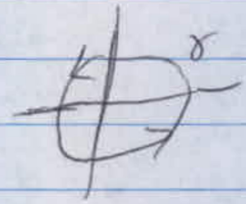
Thus $A = \iint_D (y^2+x^2+1)^{1/2}\, dA$

$$= \int_0^{2\pi} d\theta \int_0^1 dr\, (r^2+1)^{1/2}\, r$$

$$= (2\pi) \quad \frac{1}{2}\frac{2}{3}(r^2+1)^{3/2}\Big|_0^1 = \boxed{\frac{2\pi}{3}\left(2^{3/2}-1\right)}$$

5) Let Σ be $x^2+y^2+z^2=16$, $z \geq 0$, oriented w/ upward normal.
Let $\bar{F} = (x^2+z)\hat{i} + 3xyz\,\hat{j} + (2xz)\hat{k}$
Compute $\int_\Sigma (\nabla \times F)\cdot d\bar{A} \equiv I$

Stokes $\Rightarrow$ $I = \int_\gamma \bar{F}\cdot d\bar{s}$ where γ is circle of radius 4 in xy plane traversed ccw viewed from above.



$\gamma(t) = (4\cos t, 4\sin t, 0)$ $t \in [0, 2\pi]$

So $I = \int_0^{2\pi} (16\cos^2 t, 0, 0)\cdot(-4\sin(t), 4\cos t, 0)\, dt$

$$= \int_0^{2\pi} -64\cos^2 t \sin t\, dt = +\frac{64}{3}\cos^3(t)\Big|_0^{2\pi} \boxed{=0}$$

6) Find flux of $F = x^2y\hat{i} + z^8\hat{j} - 2xyz\hat{k}$ out of surface of std. unit cube $(0 \leq x, y, z \leq 1)$ in $\mathbb{R}^3$

Divergence Thm $\iiint_V (\nabla \cdot F)\, dV = \iint_S F \cdot d\bar{S} \equiv I$

$\nabla \cdot F = 2xy + 0 - 2xy = 0$

So $I = 0$

7) Find $\int_\gamma \bar{F} \cdot d\bar{s}$ w/ $F = x\hat{i} + y\hat{j} + z\hat{k}$ and γ is oriented curve given by $\gamma = (\sin^2 t \cos t,\ \cos^2 t \sin t,\ (t-\pi)^4)$ $t \in [0, 2\pi]$

method i) use stokes, γ is closed, so look for S s.t. $\partial S = \gamma$

Compute $\nabla \times F = 0$, So $I = 0$

ii) $\int_\gamma \bar{F} \cdot d\bar{s} = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \cdots$

8) $F = 3x^2y\hat{i} + x^3\hat{j} + 5\hat{k}$

$\bar{G} = (x+z)\hat{i} + (z-y)\hat{j} + (x-y)\hat{k}$

$$\nabla \times G = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x+z & z-y & x-y \end{vmatrix} = \hat{x}(-1-1) - \hat{y}(1-1) + \hat{z}(0) \neq 0$$

Not conservative

$\nabla \times F = 0$, Conservative

$F = \nabla f$; $\quad \frac{\partial f}{\partial x} = 3x^2 y \Rightarrow f(x,y,z) = x^3 y + g(y,z)$

$$\frac{\partial f}{\partial y} = x^3 \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = g(z)$$

$$\frac{\partial f}{\partial z} = 5 \Rightarrow \frac{\partial g}{\partial z} = 5 \Rightarrow g = 5z + C$$

$\boxed{\text{So } f = x^3 y + 5z + C}$