20E Spring 2012 Lecture B Final Solution

1 By Green Theorem, let D be the region Contained in Γ

$$\int_\Gamma (4y-3x)\,dx + (x-4y)\,dy$$

$$= \int_D \frac{\partial\, (x-4y)}{\partial x} - \frac{\partial\, 4y-3x}{\partial y}\, dx\,dy$$

$$= \int_D 1 - 4\, dx\,dy = -6\pi$$

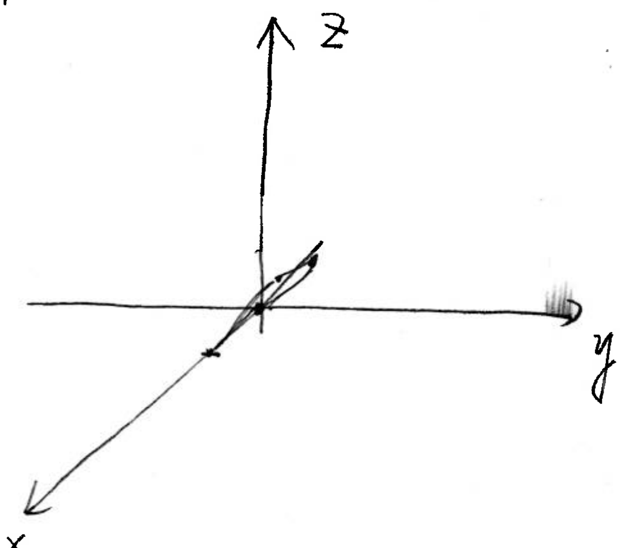
2 First find parametrization of Γ by let $t=x$

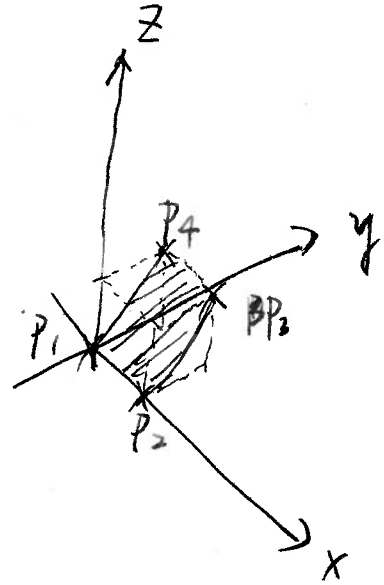
$$c(t) = (t, 0, t^2) \quad t \in [-1, 2]$$

$$\int_\Gamma F\cdot ds = \int_{-1}^{2} F(t)\cdot c'(t)\, dt = \int_{-1}^{2} \begin{pmatrix} 0 \\ t \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2t \end{pmatrix} dt$$

$$= \int_{-1}^{2} 2t^3\, dt = \left.\frac{t^4}{2}\right|_{-1}^{2} = 7.5$$

3.





First, we want to find formula for this surface. To fix a linear surface, we need it's normal vector. We know:

$$\begin{cases} \vec{n} \perp P_1 - P_2 \\ \vec{n} \perp P_3 - P_2 \\ \vec{n} \perp P_3 - P_4 \end{cases}$$
$$\vec{n} \perp P_4 - P_1$$

use any 3 of them. We have a normal vector: $\vec{n} = (0, -1, 1)$ (usually we want normal vector with length 1. but this time we only want direction, so doesn't matter)

$\therefore$ formula for this plane is $0 \cdot x + (-1) y + 1 \cdot z = K$

plug in P_1, we know $K = 0 \Rightarrow -y + z = 0$

(you can plug in P_2, P_3, P_4 to verify this formula is correct!)

Then we can parametrize this plane by $\begin{cases} u = x \\ v = y \end{cases}$.

$$T(u,v) = (\cancel{x,y}, (u, v, v) \qquad \begin{cases} u \in [0,1] \\ v \in [0,1] \end{cases}$$

$$T_u \times T_v = \begin{pmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = (0, -1, 1) \text{ (same direction with } \vec{n})$$

$$\| T_u \times T_v \| = \sqrt{2}$$

$$\int_R xyz\, dS = \int_0^1\int_0^1 uv^2\cdot\sqrt{2}\, du\, dv = \frac{\sqrt{2}}{6}$$

4. $T(u,v)$ is given,

$$T_u \times T_v = \begin{pmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{pmatrix} = \left(-2u^2\cos v,\ -2u^2\sin v,\ u\cos^2 v + u\sin^2 v\right)$$

$$\|T_u \times T_v\| = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2}$$

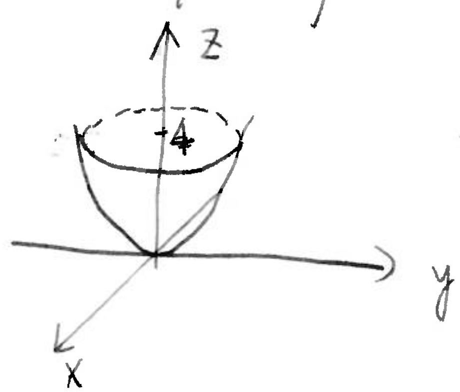
$$= \sqrt{4u^4+u^2}$$

$$\text{Area} = \int_0^{2\pi}\int_0^2 \sqrt{4u^4+u^2}\, du\, dv$$

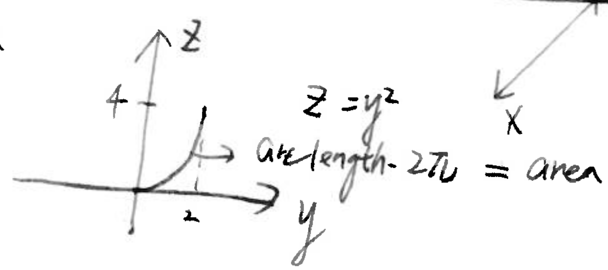
$$= \int_0^{2\pi}\int_0^2 u\sqrt{4u^2+1}\, du\, dv$$

$$= \int_0^{2\pi}\int_0^2 \frac{\sqrt{4u^2+1}}{2}\, du^2\, dv = \frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

Actullay if you can observe the patten of the surface you can let $z = u^2 = r^2$



Use rotation to calculate area



5. $T(u,v)$ is given.

$$T_u \times T_v = \begin{pmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{pmatrix} = (\sin v, -\cos v, u)$$

$$F(u,v) = (u\sin v, -u\cos v, v^3)$$

$$\int_\Sigma F(u,v) \cdot T_u \times T_v \; du\,dv$$

$$= \int_0^{2\pi}\int_0^2 u + uv^3 \; du\,dv = 4\pi + 8\pi^4$$

6. By Gauss's Theorem

$$\iiint_B \operatorname{div}(F)\, dV = \iint_{\partial B} F \, d\vec{S}$$

$$\operatorname{div} F = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_B 3x^2 + 3y^2 + 3z^2 \; dx\,dy\,dz$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \cdot \rho^2 \sin^2\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \frac{3}{10}\pi^2$$

7. You need observe $(\sin^2 0, \cos^3 0, \sin^4 0) = (\sin^2 2\pi, \cos^2 2\pi, \sin^4 2\pi)$

So it is nature to think ~~Do~~ is there a function G, s.t

$\nabla G = F$ or does $\nabla \times F = 0$ (if so, when we use Green ~~to~~ theorem, we don't need to deal with the complex boundary)

Actullay G ~~[illegible]~~ exist, $G = \frac{1}{2}x^2 + \frac{y^3}{3} + \frac{z^4}{4}$. So by fundamental law of the integral, $\int_r F\, ds = 0$

8. The defination of Conservtive is on page 453 of textbook.

But I guess that will not be cover in your final.

You can compute $\nabla \times F$ and $\nabla \times G$ see which one is zero.

OR: for $G := \underbrace{(x^3-3xy^2)}_{P} i + \underbrace{(y^3-3x^2y)}_{Q} j + \underbrace{z}_{R} k$

$$\int P\,dx = \frac{x^4}{4} - \frac{3x^2y^2}{2} + f_1(y,z)$$

$$\int Q\,dy = \frac{y^4}{4} - \frac{3x^2y^2}{2} + f_2(x,z) \Rightarrow G = \nabla f$$

$$\int R\,dz = \frac{z^2}{2} + f_3(x,y)$$

with $f = \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + \frac{z^2}{2} + C$