4.2) Null space and column space

Given any $m\times n$ matrix A, there are two important subspaces: $(A=[\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n])$.

1) The null space of A:

$$\text{Nul}(A) = \{\vec{x}\in \mathbb{R}^n : A\vec{x}=\vec{0}\}.$$

= the solution set of the homogeneous system whose coefficient matrix is A.

2) The column space of A:

$$\text{Col}(A) = \text{span}\{\vec{a}_1,\dots,\vec{a}_n\}.$$
$$= \{A\vec{x} : \vec{x}\in \mathbb{R}^n\}.$$

Thm: For ~~A~~ an $m\times n$ matrix A,

Nul(A) is a subspace of $\mathbb{R}^n$.

Col(A) is a subspace of $\mathbb{R}^m$.

Pf: Exercise.

E.g. Find the null space of $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$.

$$\text{RREF}(A)=\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}.$$

x_3 = free

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\-2x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}. \Rightarrow \text{Null}(A)=\text{span}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}.$$

(40)

Q: Is the solution set $\underset{\overset{\parallel}{W}}{\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b}\}}$ a subspace if $\vec{b} \neq \vec{0}$?

A: No! $\vec{0} \notin W$ as $A\vec{0} = \vec{0} \neq \vec{b}$.

4.3) Linearly Independence & Bases.

. vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in the same vector space V are <u>linearly dependent</u> if one of them is a linear combination of the others.

~~$\Leftrightarrow$ there is a nontrivial linear combination of them~~

($\Leftrightarrow$ the system

$$[\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$ has a <u>nontrivial</u> solution.)

$\Leftrightarrow$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = 0 \Rightarrow$ has a nontrivial solution

. vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ <u>linearly independent</u> if they are <u>not</u> linearly dependent.

($\Leftrightarrow$ $[\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]\begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$.)

$\Leftrightarrow$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0} \Rightarrow x_1 = x_2 = \ldots = x_n = 0$.

<u>E.g.</u> 1) $\mathbb{P} = \{\text{all polynomials}\}$.

$\vec{v}_1 = x^2 + 1$, $\vec{v}_2 = -x$, $\vec{v}_3 = -x^2 + x - 1$.

These vectors are linearly dependent.

$$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0.$$

2) $\vec{v}_1 = x^2 + 1$ $\quad \vec{v}_2 = -x$ $\quad \vec{v}_3 = 1$.

Consider

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = 0.$$

$$\alpha_1(x^2+1) + \alpha_2(-x) + \alpha_3 = 0$$

$$\alpha_1 x^2 - \alpha_2 x + \alpha_1 + \alpha_3 = 0.$$

$\Rightarrow \alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$. $\Rightarrow$ these vectors are linearly independent.

Def: Let V be a vector space, and let $H \subseteq V$ be a subspace.

A collection $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ of vectors in H is called a <u>basis</u> for H if

- $H = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent.

<u>E.g.</u> The "standard basis vectors"

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

are a basis for $\mathbb{R}^n$.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

every row is pivotal
column is pivotal.

<u>Fact:</u> $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$;
$\iff [\vec{v}_1 \ \vec{v}_2 \dots \vec{v}_n]$ is invertible.

<u>E.g.</u> Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Yes!

~~Thm~~:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 6 & -1 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \text{pivots} \\ \leftarrow \end{array} \\ \uparrow \quad \uparrow \quad \uparrow \qquad \\ \text{pivots} \qquad \end{array}$$

Thm: Any basis of $\mathbb{R}^n$ consists of exactly n column vectors. The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis $\Leftrightarrow$ the matrix $[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$ is invertible.

E.g. Let $W \subseteq \mathbb{R}^3$ be the set of all vectors of the form

$$\begin{bmatrix} a+b \\ a-b \\ a \end{bmatrix}. \text{ Then } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } W.$$

Subspaces are frequently presented as the span of some collection of vectors. But those vectors may not be linearly independent.

E.g. $\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

But $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is <u>linearly dependent</u>.

E.g. Consider

$$H = \text{span}\left\{ \underset{\vec{u}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}} \right\}.$$

since $\vec{w} = \vec{u} - \vec{v}$, $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent.

Q: What is a basis for H?

For any $\vec{x} \in H$, there are some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$$\begin{aligned} \vec{x} &= \alpha_1 \vec{u} + \alpha_2 \vec{v} + \alpha_3 \vec{w} \\ &= \alpha_1 \vec{u} + \alpha_2 \vec{v} + \alpha_3 (\vec{u} - \vec{v}) \\ &= (\alpha_1 + \alpha_3)\vec{u} + (\alpha_2 - \alpha_3)\vec{v}. \end{aligned}$$

$\Rightarrow \vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$.

$\therefore H = \text{span}\{\vec{u}, \vec{v}\}$.

Thm: If $\mathcal{H} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$, then some collection of these vectors is a basis for $\mathcal{H}$.

Thm: The pivotal columns of A form a basis for Col(A) (The coefficients expressing the non-pivotal columns in terms of this basis can be read off of RREF(A)).

E.g.
$$\begin{bmatrix} \boxed{1} & \boxed{3} & 2 \\ \boxed{0} & \boxed{-1} & -1 \\ \boxed{2} & \boxed{1} & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\} \text{ form a basis for Col}(A).$$

Remark: The pivotal columns of A, <u>not REF(A)</u>, form a basis for Col(A). Typically

$$\text{Col}(A) \neq \text{Col}(\text{REF}(A)).$$

E.g.
$$\mathcal{H} = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\} = \text{Col}(A).$$

Find a basis for Col(A).

So we know how to find a basis of $Col(A)$.
How about $Nul(A)$?

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

pivots pivot.

$$Nul(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

linearly independent.

The above process for identifying $Nul(A)$ always produce a basis for it.

Remark: A given subspace always has many bases, but ~~any two bases~~

Thm: Any two basis for a subspace have the same number of vectors.

E.g. $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = W$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ basis for W.

$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$, but $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ is not a basis for W (why?).

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for W.

<u>Def:</u> If V is a vector space, its <u>dimension</u> is the numbers of vectors in any basis.

E.g. $V = \text{span}\left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$.

$$A = \begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\-2 & 2 & 1\end{bmatrix}$$ has 3 pivotal columns.

$$\dim(V) = 3.$$

E.g. $\mathbb{P}_3 = \{$ all polynomials of degree $\leq 3\}$.

$= \text{span}\{x^3, x^2, x, 1\}$.

$\Rightarrow \dim(\mathbb{P}_3) = 4$.

The <u>dimension</u> of a vector space V is the number of vectors in any basis.

<u>E.g.</u> $V = \left\{ \begin{bmatrix} a+b \\ a-b \\ a \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$.

$$= \operatorname{span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \right\}$$

$\vec{v}_1$ $\vec{v}_2$ form a basis for V.

$\dim V = 2$.

Other base $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

<u>Def:</u> Given a matrix A,

$$\operatorname{rank}(A) := \dim(\operatorname{Col}(A))$$
$$\operatorname{nullity}(A) := \dim(\operatorname{Nul}(A)).$$

$$A = \begin{bmatrix} 1 & -3 & 6 \\ 3 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

pivots

form a basis for $\operatorname{Col}(A)$.

$\Rightarrow \operatorname{rank}(A) = 2$

$$\operatorname{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} \Rightarrow \operatorname{nullity}(A) = 1.$$

Thm: The pivotal columns of A form a basis for $Col(A)$

$$\text{rank}(A) = \text{number of pivotal columns of } A.$$
$$\text{nullity}(A) = \text{number of free variables.}$$
$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A.$$

Def: The row space of A, $Row(A)$, is the span of the rows of A.

E.g. $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$Row(A) = \text{span}\{[-2\ -5\ 8\ 0\ -17], [1\ 3\ -5\ 1\ 5], [3\ 11\ -19\ 7\ 1], [1\ 7\ -13\ 5\ -3]\}$$
$$= \text{span}\{[1\ 3\ -5\ 1\ 5], [0\ 1\ -2\ 2\ -7], [0, 0\ 0\ -4\ 20]\}.$$

Thm: $Row(A) = Row(RREF(A))$.

Thm: A $n\times n$ matrix. The following statements are each equivalent to "A is an invertible matrix."

1) The columns of A form a basis of $\mathbb{R}^n$.
2) $Col(A) = \mathbb{R}^n$ 3) $\dim(Col(A)) = n$
4) $\text{rank}(A) = n$
5) $Nul(A) = \{0\}$ 6) $\dim(Nul(A)) = 0$.

4.4) Coordinates.

Thm: If $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for V, then each vector $\vec{v} \in V$ has a unique expansion.

$$(*) \quad \vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \dots + x_n\vec{b}_n.$$

for some unique scalars $x_1, \dots, x_n \in \mathbb{R}$.

Pf: Suppose ~~there~~ $\vec{v}$ has ~~is~~ a different expansion from (*),

$$(**) \quad \vec{v} = y_1\vec{b}_1 + y_2\vec{b}_2 + \dots + y_n\vec{b}_n.$$

then (*) − (**) implies

$$\vec{v} - \vec{v} = (x_1 - y_1)\vec{b}_1 + (x_2 - y_2)\vec{b}_2 + \dots + (x_n - y_n)\vec{b}_n$$

$$\vec{0} = (x_1 - y_1)\vec{b}_1 + (x_2 - y_2)\vec{b}_2 + \dots + (x_n - y_n)\vec{b}_n.$$

Since $\{\vec{b}_1, \dots, \vec{b}_n\}$ linearly independent,

$$\begin{aligned} x_1 - y_1 &= 0 \\ x_2 - y_2 &= 0 \\ &\vdots \\ x_n - y_n &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ &\vdots \\ x_n &= y_n. \end{aligned}$$

This means if V has a basis $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ (note $\dim(V) = n$) then we can identify V with $\mathbb{R}^n$ by identifying each vector $\vec{v}$ with its B-coordinate vector. i.e.,

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \dots + x_n\vec{b}_n$$

$$\rightsquigarrow [\vec{v}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

E.g. $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, then its coordinate vector with respect to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is just $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

But what if we use the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

E.g. $\mathbb{P}_4 = \{$ polynomials of degree $\leq 4 \} = \{ ax^4 + bx^3 + cx^2 + dx + f ; a, b, c, d, f \in \mathbb{R} \}$.

"standard basis" $\cancel{\mathcal{B} = \{x^4, x^3, x^2, x, 1\}}$.

$\mathcal{B} = \{1, x, x^2, x^3, x^4\}$.

Consider $p(x) = \cancel{(x-1} \; x^3 - 3x + 1$.

$\cancel{\Rightarrow [p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \\ 4 \end{bmatrix}}$. $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

Thm: If $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V then the function $T : V \to \mathbb{R}^n$ defined by $T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Such a linear transformation is called an isomorphism.

E.g. in $\mathbb{P}_4$, a typical element $p(x) \in \mathbb{R}.\mathbb{P}_4$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

standard basis $\mathcal{B} = \{1, x, x^2, x^3, x^4\}$.

$$[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

The map $T(p) = [p]_{\mathcal{B}}$ is an isomorphism from $\mathbb{P}_4$ onto $\mathbb{R}^5$.

Remark: isomorphisms preserve all linear properties.

Thm: Let $\{\vec{b}_1, \dots, \vec{b}_n\} = \mathcal{B}$ be a basis for V.

Then 1) $\{\vec{v}_1, \dots, \vec{v}_k\} \in V$ are linearly independent in V
$\iff \{[\vec{v}_1]_{\mathcal{B}}, [\vec{v}_2]_{\mathcal{B}}, \dots, [\vec{v}_k]_{\mathcal{B}}\}$ are linearly independent in $\mathbb{R}^n$.

2) $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans V
$\iff \{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_k]_{\mathcal{B}}\}$ spans $\mathbb{R}^n$.

E.g. Show that $\{1, x-1, (x-1)^2\}$ is a basis for $\mathbb{P}_2 = \{$all polynomials of deg $\leq 2\}$.

Standard basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{P}_2$.

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [x-1]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad [(x-1)^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

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$$\begin{bmatrix} \boxed{1} & -2 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

these vectors form a basis for $\mathbb{R}^3$.

$\Rightarrow$ $\{1, (x-1), (x-1)^2\}$ form a basis for $\mathbb{P}_2$.

3.1) Determinants

Recall that for a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

its determinant is $ad-bc$

we denote $\det(A) = |A|$.

A is invertible iff $\det(A) \neq 0$

in which case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Q: What is the determinant of an nxn matrix A.

For any nxn matrix A, there is a quantity det A which

1) is a polynomial function of entries of A
2) determines whether A is invertible.

That is det(A), determinant of A.

* Cofactor Expansion.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \end{aligned}$$

$\Big\}$ True for any row or column j.

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$$C_{ij} \text{ is } (i,j)\text{-cofactor of } A \text{ given by.}$$
$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

where A_{ij} is the submatrix formed by deleting the ith row and jth column of A.

E.g. $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$. Find $\det A$?

$$\det A = 1 \cdot C_{11} + 5C_{12} + 0 \cdot C_{13}.$$

$$C_{11} = (-1)^2 \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} = 1.\,(4 \cdot 0 - 2) = -2.$$

$$C_{12} = (-1)^3 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = 0.$$

$$C_{13} = (-1)^4 \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} = 1.(2.(-2) - 0.4) = -2.$$

$$\therefore \det A = 1(-2) + 5(0) + 0(-2) = -2.$$

Alternatively, we can compute $\det A$ as follows.

$$\det A = a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32}.$$

$$= 5(-1)^3 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 4(-1)^4 \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)(-1)^5 \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= -1(0) + 1(0) + (-2)(-1)(-1)$$
$$= -2.$$

Thm: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Why?
$$\begin{bmatrix} x & & & \\ 0 & x & & \\ 0 & 0 & x & \\ 0 & 0 & 0 & x \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} = 1 \cdot (-1)^2 \det \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$
$$= 1 \cdot 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$
$$= 1 \cdot 3 \cdot 1 \cdot 5.$$

3.2) Properties of Determinants.

* Determinants & Row operations.

- $\det \begin{bmatrix} k(-R_1-) \\ -R_2- \\ -R_3- \end{bmatrix} = k \det \begin{bmatrix} -R_1- \\ -R_2- \\ -R_3- \end{bmatrix}$

- $\det \begin{bmatrix} -R_1- \\ -kR_1+R_2- \\ -R_3- \end{bmatrix} = \det \begin{bmatrix} -R_1- \\ -R_2- \\ -R_3- \end{bmatrix}$

- $\det \begin{bmatrix} -R_1- \\ -R_2- \\ -R_3- \end{bmatrix} = - \det \begin{bmatrix} -R_2- \\ -R_1- \\ -R_3- \end{bmatrix}$

$|\cdot| = \det(\cdot)$ another notation for determinant

E.g. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot(-1)^3 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^5 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Thm: A square matrix A is invertible $\Leftrightarrow$ $\det(A) \neq 0$.

Thm: • If A is $n \times n$ and has full rank (i.e., $\operatorname{rank}(A) = n$, A invertible) then $\det A = \pm$ product of the pivots in row reduction.

depends on how many times we interchange the rows

• If $\operatorname{rank}(A) < n$, $\det A = 0$.

E.g.

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \xrightarrow[(-1)]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$$

$\det(\;) = 13$

$\Rightarrow \det A = (-1) \cdot 13 = -13.$

* Elementary Matrices:

An elementary matrix is the result of one ~~w~~ row operation applied to the identity matrix.

E.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cancel{2}R_2 \to 2R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\cancel{0}$ 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thm: If π is a row operation and $I \xrightarrow{\pi} E$, then for any matrix A, if $A \xrightarrow{\pi} \tilde{A}$,
$$\tilde{A} = EA.$$

E.g.
$$\overset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}} \xrightarrow{R_2 - 2R_1} \overset{\tilde{A}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$\overset{E}{\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}} \overset{\checkmark}{=} \overset{\tilde{A}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix}}$$

Recall that if A is invertible, $RREF(A) = I$.

$$\begin{array}{ccccccc} A & \xrightarrow[E_1]{\pi_1} & A_1 & \xrightarrow[E_2]{\pi_2} & A_2 & \xrightarrow[E_3]{\pi_3} & \cancel{A_3}\ I \\ & & \| & & \| & & \| \\ & & E_1 A & & E_2 A_1 & & E_3 A_2 . \end{array}$$

$$\therefore \quad I = (E_3 E_2 E_1) A.$$
$$(E_3 E_2 E_1)^{-1} I = A$$
$$E_1^{-1} E_2^{-1} E_3^{-1} = A.$$

Thm: If A, B are $n \times n$ matrices,
$$\det(AB) = \det(A)\det(B).$$

Remark: $\det(A+B) \neq \det(A) + \det(B)$ in general.

3.3) Properties of determinants; Volumes.

- $\det(A^{-1}) = \dfrac{1}{\det(A)}$ if A is invertiable.

Pf: $AA^{-1} = I$.

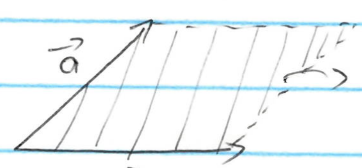
$\det(AA^{-1}) = \det(I)$

$\det(A)\det(A^{-1}) = 1$.

- $\det(A^T) = \det(A)$.

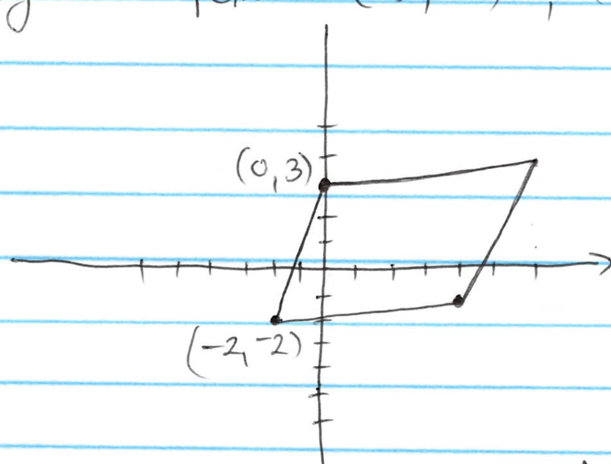
Thm: Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$, the area of the parallelogram they determine is

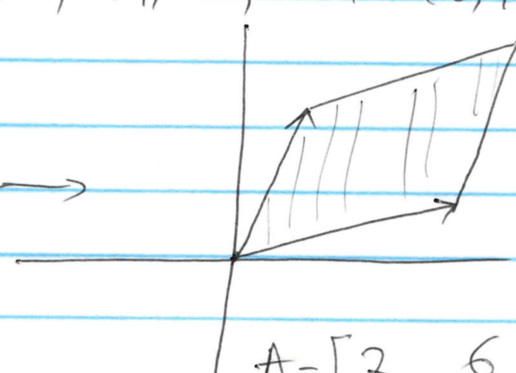
$$A(\vec{a}, \vec{b}) = \left|\det\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \vec{a}^T \\ \vec{b}^T \end{bmatrix}\right|.$$



$= |\det[\vec{a}, \vec{b}]|$.

E.g. Calculate the area of the parallelogram determined by the points $(-2,-2)$, $(0,3)$, $(4,-1)$, and $(6,4)$.

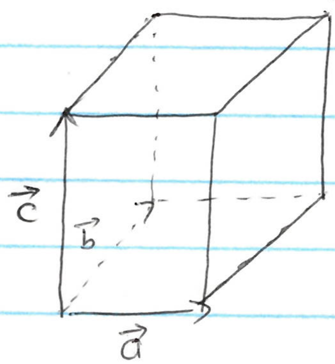




$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}.$$

$\rightarrow$ Area $= |\det(A)| = |2 - 30| = 28$.

What about $n \times n$ determinants with $n > 2$?



$$\text{Volume}(P) = \left| \det \left[\vec{a} \ \vec{b} \ \vec{c} \right] \right|.$$