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2.1) Matrix Operations.

We can treat matrices of the same size like vectors and <u>add</u> them and <u>multiply</u> them by scalars.

<u>E.g.</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \end{bmatrix}$.

$$A + B = \begin{bmatrix} 2 & 2 & 4 \\ 3 & 7 & 9 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 2 \\ -2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 6 & 1 & 0 \end{bmatrix}.$$

But if $C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, we cannot add A to C.

$\cancel{A + C}$

<u>Properties</u> (like real numbers).

Suppose A, B, C are matrices of same size.

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + \underbrace{0}_{\text{zero matrix}} = A$$

$$r(A + B) = rA + rB \quad , \; r \in \mathbb{R}.$$

$$(r + s)A = rA + sA \quad , \; r \text{ and } s \in \mathbb{R}.$$

$$r(sA) = (rs)A.$$

Matrix addition and scalar multiplication correpond to addition and scalar multiplication of linear transformations. That is, if $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations and are defined by

$$T(\vec{x}) = A\vec{x} \qquad \text{and} \qquad S(\vec{x}) = B\vec{x}.$$

then

$$(T+S)(\vec{x}) = T(\vec{x}) + S(\vec{x}) = A\vec{x} + B\vec{x} = (A+B)\vec{x}.$$

$$(rT)(\vec{x}) = rT(\vec{x}) = rA\vec{x} = (rA)\vec{x}.$$

* <u>Matrix multiplication:</u>

If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^k$ are linear, we can compose them.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

$S\circ T: \mathbb{R}^n \to \mathbb{R}^k$ $\qquad (S\circ T)(\vec{x}) = S(T(\vec{x})).$

<u>Facts:</u> 1) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$:

$$(S\circ T)(\vec{u} + \vec{v}) = (S\circ T)(\vec{u}) + (S\circ T)(\vec{v}).$$

<u>proof:</u> $\text{LHS} = (S\circ T)(\vec{u} + \vec{v})$

$$= S(T(\vec{u} + \vec{v})).$$

$$= S(\underbrace{T(\vec{u})}_{\vec{x}} + \underbrace{T(\vec{v})}_{\vec{y}}).$$

$$= S(\vec{x}) + S(\vec{y})$$

$$= (S\circ T)(\vec{u}) + (S\circ T)(\vec{v}).$$

2) For any $\vec{u} \in \mathbb{R}^n$ and any scalar α:

$$(S\circ T)(\alpha\vec{u}) = \alpha(S\circ T)(\vec{u}).$$

$\Rightarrow$ $S\circ T$ is linear.

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $S: \mathbb{R}^m \to \mathbb{R}^k$.

$T(\vec{x}) = A\vec{x}$ $\qquad$ $S(\vec{y}) = B\vec{y}$.

A standard matrix of T $\qquad$ B standard matrix of S.

Q: What is the standard matrix of $(S \circ T)$?

$$(S\circ T)(\vec{x}) = S(T(\vec{x})) = S(A\vec{x})$$

$A = [\vec{a}_1\ \vec{a}_2 \ldots \vec{a}_n]$

$$= S(x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n)$$
$$= x_1 S(\vec{a}_1) + x_2 S(\vec{a}_2) + \ldots + x_n S(\vec{a}_n)$$
$$= x_1 B\vec{a}_1 + x_2 B\vec{a}_2 + \ldots + x_n B\vec{a}_n$$
$$= \underbrace{[B\vec{a}_1\ \ B\vec{a}_2\ \ldots\ B\vec{a}_n]}_{\text{Standard matrix of } S\circ T}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

<u>Def:</u> If A is $m \times n$ matrix and B $k \times m$, the product BA is defined by
$$BA = B[\vec{a}_1\ \vec{a}_2 \ldots \vec{a}_n] := [B\vec{a}_1\ \ B\vec{a}_2 \ldots B\vec{a}_n].$$

<u>E.g.</u>
$$\underset{3\times 2}{\overset{B}{\begin{bmatrix} 4 & 1 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}}}\ \underset{2\times 2}{\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}} = \begin{bmatrix} B\begin{bmatrix}1\\3\end{bmatrix} & B\begin{bmatrix}2\\4\end{bmatrix}\end{bmatrix}$$
$$= \begin{bmatrix} 4\cdot 1 + 1\cdot 3 & 4\cdot 2 + 1\cdot 4 \\ -2\cdot 1 - 1\cdot 3 & -2\cdot 2 - 1\cdot 4 \\ 0\cdot 1 + 1\cdot 3 & 0\cdot 2 + 1\cdot 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 12 \\ -5 & -8 \\ 3 & 4 \end{bmatrix}$$

E.g. $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}4 & 1\\ -2 & -1\\ 0 & 1\end{bmatrix}$ does not make sense.

$2\times\boxed{2 \qquad 3}\times 2$

~~To appl~~ To multiply two matrices, the number of columns of matrix on the left have to be the number of rows of matrix on the right.

* <u>Properties of matrix multiplication</u>:

- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$
- $(rA)B = r(AB) = A(rB)$.

If A is $m\times n$, $I_m A = A I_n = A$.

$m\times m$ identity matrix $\qquad$ $n\times n$ identity matrix.

<u>Remark</u>: real numbers ~~are~~ commute:

$$ab = ba \quad \text{for } a, b \in \mathbb{R}.$$

But this may not hold for matrices.

E.g. $A = \begin{bmatrix}4 & 7\\ 3 & 3\end{bmatrix}$ $\quad B = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$.

$AB = \begin{bmatrix}11 & 3\\ 6 & 0\end{bmatrix}$ but $BA = \begin{bmatrix}1 & 4\\ 7 & 10\end{bmatrix}$.

* Transpose:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

$m \times n$ $\qquad$ $m \times n$.

If A is $m \times n$, the A^T is $n \times m$

$(A^T)^T = A$ $\qquad$ $(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$ $\qquad$ $(rA)^T = r A^T$.

where $r \in \mathbb{R}$.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \Rightarrow \vec{v}^T = [v_1 \ v_2 \ \dots \ v_m].$$

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n] \Rightarrow A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}.$$

2.2 - 2.3) Matrix Inverse

Suppose we want to solve $ax = b^{\neq 0}$ for x.

If $a = 0$, no solution.

If $a \neq 0$,

$$ax = b$$
$$\frac{ax}{a} = \frac{b}{a}$$
$$a^{-1}ax = a^{-1}b$$
$$x = a^{-1}b.$$

Q: Can we solve $A\vec{x} = \vec{b}$ the same way?

Def: Let A be an $\underset{\text{(square)}}{n \times n}$ matrix. It is called <u>invertible</u> if there is a matrix C satisfying

$$CA = I_n = AC.$$

Then we denote $C = A^{-1}$.

Thm: A is invertible ~~if~~ $\overset{(\text{iff})}{\Longleftrightarrow}$ all rows and columns are pivotal.

$\Longleftrightarrow \text{RREF}(A) = I_n$.

Note: Invertible matrices must be square.

- $AA^{-1} = I$ if $A\vec{x} = \vec{b}$ is solvable for any $\vec{b}$
  $\Longleftrightarrow$ every row is pivotal.
- $A^{-1}A = I$ if $A\vec{x} = \vec{b}$ has a unique solution whenever it is consistent,
  $\Longleftrightarrow$ every column is pivotal.

Q: How do we find the inverse, if it exists?

Looking for the matrix $C = [\vec{c}_1\ \vec{c}_2 \dots \vec{c}_n]$ such that $AC = I$.

$$A[\vec{c}_1\ \vec{c}_2 \dots \vec{c}_n] = [\vec{e}_1\ \vec{e}_2 \dots \vec{e}_n]$$

$$[A\vec{c}_1\ A\vec{c}_2 \dots A\vec{c}_n] = [\vec{e}_1\ \vec{e}_2 \dots \vec{e}_n]$$

$\therefore$
$$[A\vec{c}_1 \mid \vec{e}_1]$$
$$[A\vec{c}_2 \mid \vec{e}_2]$$
$$\vdots$$
$$[A\vec{c}_n \mid \vec{e}_n]$$

We can consider

$$[A \mid I] \xrightarrow{RREF} [I \mid A^{-1}]$$

That is, to find the inverse of A, consider $[A \mid I]$ and perform row operations until ~~A~~ we get RREF. If $RREF(A) = I$, then the right hand side is A^{-1}.

E.g. Find the inverse of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

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E.g.
$$\begin{array}{cc} A & C \\ \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} & \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} C & A \\ \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Relation to linear transformations:

Standard matrix A and its linear transformation $T(\vec{x}) = A\vec{x}$

- $A\vec{x} = \vec{b}$ solvable for all $\vec{b}$ $\Leftrightarrow$ T is onto.
- $A\vec{x} = \vec{b}$ has at most one solution for each $\vec{b}$ $\Leftrightarrow$ T is one-to-one.

Thm: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto.

An $n \times n$ A is invertiable

$\Leftrightarrow$ A has n pivot positions.

$\Leftrightarrow$ there exists $n \times n$ C with $CA = I$.

$\Leftrightarrow$ there exists $n \times n$ D with $AD = I$.

$\Big\}$ in both cases $C = D = A^{-1}$.

Note: Not all square matrices are invertible!

E.g. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

E.g. General 2x2 matrix

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[R_2 a]{R_1 c} \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right]$$

$a \neq 0$

$c \neq 0$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

Let $D = ad-bc$

"determinant"

$$\xrightarrow[\text{if } D \neq 0]{R_2/D} \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

$$\xrightarrow{R_1 - bc \cdot R_2} \left[\begin{array}{cc|cc} ac & 0 & c + \frac{c^2 b}{D} & -\frac{abc}{D} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

$$\xrightarrow{R_1/ac} \left[\begin{array}{cc|cc} 1 & 0 & \frac{c}{ac} + \frac{c^2 b}{Dac} & -\frac{abc}{Dac} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{D} & \frac{-b}{D} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right].$$

Thm: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff$ $ad - bc \neq 0$

in this case $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

E.g. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \dfrac{1}{1.3 - 1.2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$

But $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

Remark: 1) $(A^{-1})^{-1} = A$.

2) $(A^{T})^{-1} = (A^{-1})^{T}$.

4.1) Vector spaces.

<u>Def:</u> A <u>vector space</u> V is a set of object, which we call <u>vectors</u>, on which two operations are defined:

addition: $\vec{u}, \vec{v} \mapsto \vec{u} + v \in V$

scalar multiplication: $\alpha, v \mapsto \alpha v \in V$ (for $\alpha \in \mathbb{R}$.)

<u>E.g.</u> $\mathbb{R}^3$, is a vector space.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} := \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \in \mathbb{R}^3$$

$$\alpha \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} := \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{bmatrix} \in \mathbb{R}^3$$

(See textbook for ~~a~~ more formal definition.)

<u>E.g.</u> $\mathbb{P} = \{$ all polynomials in a single variable $\}$.

consider

$$p(x) = x^3 + x^2 - 2x \qquad \deg(P) = 3$$

$$q(x) = x^2 - 1 \qquad \deg(q) = 2.$$

then,

$$p(x) + q(x) = (x^3 + x^2 - 2x) + (x^2 - 1) = x^3 + 2x^2 - 2x - 1 \in \mathbb{P}.$$

$$(3p)(x) = 3(x^3 + x^2 - 2x) = 3x^3 + 3x^2 - 6x \in \mathbb{P}.$$

$= \{ ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R} \}$

E.g. $\mathbb{P}_3 = \{$all polynomials of degree $\leq 3\}$ is a vector space.

But $\mathbb{P} = \{$ all polynomials of degree $= 3\}$ is <u>not</u> a vector space.

$p(x) = x^3 + x$

$q(x) = -x^3$

$p(x) + q(x) = x^3 + x - x^3 = x \notin \mathbb{P}$.

E.g. $M_{2\times 3} = \{$ set of all 2×3 matrices$\}$. is a vector space.

$$\alpha \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \end{bmatrix} \in M_{2\times 3}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \end{bmatrix} \in M_{2\times 3}$$

<u>Def:</u> If V is a vector space, and $W \subseteq V$ is nonempty, W is a <u>subspace</u> of V if W is closed under addition and scalar multiplication (in V).

i.e. if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$

and if $\alpha \in \mathbb{R}$ and $\vec{w} \in W$, then $\alpha \vec{w} \in W$

and $\vec{0} \in W$.

E.g. $\{\vec{0}\}$ the trivial subspace.

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E.g. $W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

$\vec{0} \in W$ ✓

$$\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} \in W.$$

$$\alpha \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ 0 \end{bmatrix} \in W.$$

But $A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is <u>not</u> a subspace of $\mathbb{R}^3$.

$\vec{0} \notin A$.

and ~~or~~ $\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2 \end{bmatrix} \notin A.$

Fact: Let V be any vector spaces, and let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in V. Then $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace of V.

E.g. Suppose $\vec{v}_1, \vec{v}_2 \in V$ (vector space). Show that $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V.

Recall $\underbrace{\text{span}\{\vec{v}_1, \vec{v}_2\}}_{W} = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$.

- $\vec{0} \in W$ (take $\alpha_1 = \alpha_2 = 0$).
- $(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) + (\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2) = (\alpha_1 + \lambda_1)\vec{v}_1 + (\alpha_2 + \lambda_2)\vec{v}_2 \in W$
- $\lambda(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = (\lambda \alpha_1)\vec{v}_1 + (\lambda \alpha_2)\vec{v}_2 \in W$.