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1.8) Linear Transformations

1.9) The Matrix of a linear transformation

Def: A <u>matrix transformation</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function given by matrix multiplication by some $m \times n$ matrix A

E.g. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$.

Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ as $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ 2x_1 + x_2 \end{bmatrix}.$$

The image of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ~~is~~ under T is:

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Yes!

Is there other $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e., $A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

x_3 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t+1 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the range of T?

i.e. all $\vec{b}$ with $T(\vec{x}) = \vec{b}$ for some $\vec{x} \in \mathbb{R}^3$.

i.e. the set of $\vec{b}$ with $A\vec{x} = \vec{b}$.

always consistent!

$\Rightarrow$ $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2 = \text{Image}(T)$.

E.g. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$. Is $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ in the range of $T(\vec{x}) = A\vec{x}$?

$$\underset{[A \,\vdots\, \vec{b}]}{\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]} \xrightarrow{\text{row ops}} \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

inconsistent.

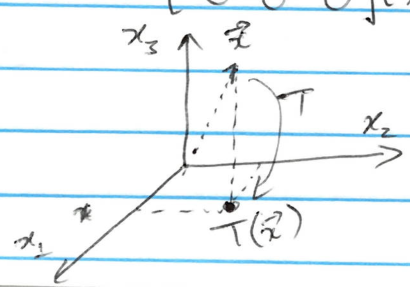
$\Rightarrow$ $\vec{b}$ is not in the range of T.

What's the range of T?

$\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} \right\}$.

* Geometric examples:

. $T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ projection onto $x_1 - x_2$ plane.



- $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.

rotation 90° counter-clockwise.

* <u>Linearity</u>: Matrix multiplication behaves well under

addition: $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$.

scalar multiplication: $A(\alpha \vec{u}) = \alpha A\vec{u}$, $\alpha \in \mathbb{R}$.

<u>Def</u>: A <u>linear transformation</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function with the properties

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$T(\alpha \vec{u}) = \alpha T(\vec{u})$ for all $\alpha \in \mathbb{R}$, and $\vec{u} \in \mathbb{R}^n$.

$\Rightarrow$ Matrix transformations are examples of linear transformations.

* <u>Properties</u>:

- $\underset{\mathbb{R}^n}{T(\vec{0})} = \underset{\mathbb{R}^m}{\vec{0}}$.
- $T(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_n \vec{u}_n) = \alpha_1 T(\vec{u}_1) + \alpha_2 T(\vec{u}_2) + \dots + \alpha_n T(\vec{u}_n)$

"superposition principle".

Def: The <u>standard basis vectors</u> in $\mathbb{R}^n$ are

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

In $\mathbb{R}^2$, there are 2 standard basis vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

In $\mathbb{R}^3$, there are 3 standard basis vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

They are the columns of the <u>identity matrix</u>.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix}$$

all entries on the diagonal $= 1$. Others are zero.

$T: \mathbb{R}^n \to \mathbb{R}^m$

Thm: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its image on the standard basis vectors.

$$T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n).$$

Moreover, every linear transformation is a matrix transformation.

E.g. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{e}_2} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_3}.$$

$$\Rightarrow T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) = x_1\underbrace{T(\vec{e}_1)}_{\vec{a}_1} + x_2\underbrace{T(\vec{e}_2)}_{\vec{a}_2} + x_3\underbrace{T(\vec{e}_3)}_{\vec{a}_3}$$

$$= \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

"The standard matrix of T:"

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}.$$

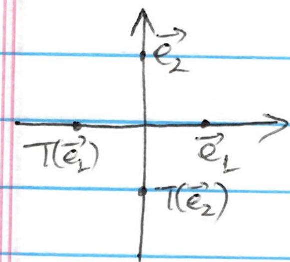
E.g. The identity function $T(\vec{x}) = \vec{x}$ is linear.

What is its matrix?

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

E.g. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a $180°$ rotation around $\vec{0}$.

(Assume it's linear). What's the matrix?



$T(\vec{e_1}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad T(\vec{e_2}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$A = [T(\vec{e_1}) \; T(\vec{e_2})] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Def: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called ~~onto if~~

- onto if its ~~whole~~ range is its codomain.
  i.e. for every $\vec{b} \in \mathbb{R}^m$, there is at least one $\vec{x} \in \mathbb{R}^n$ with $T(\vec{x}) = \vec{b}$.
- one-to-one if $T(\vec{x}) = \vec{b}$ has at most one solution for any $\vec{b}$.

E.g. $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \vec{x}$ is not one-to-one (why?)

E.g. $T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \vec{x}$ is one-to-one but not onto ? (why?)

Thm: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff:
the rows of A are pivotal
i.e. $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$.
i.e. columns of A span $\mathbb{R}^m$.

b) T is one-to-one iff: columns of A are all pivotal
i.e. columns of A are lin. independent