1.4) The matrix equation $A\vec{x} = \vec{b}$

* <u>Matrix multiplication:</u>

Let $A = [\underset{\substack{\uparrow \\ \text{1st col}}}{\vec{a}_1} \ \underset{\substack{\uparrow \\ \text{2nd col.}}}{\vec{a}_2} \ \dots \ \underset{\substack{\uparrow \\ \text{nth column}}}{\vec{a}_n}]$ be an $m \times n$ matrix.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ be a column vector,

<u>Note:</u> # of entries of $\vec{x}$ = # of columns of A.

$$A\vec{x} = \underbrace{[\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n]}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} := x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n.$$

E.g.
$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+14 \\ 8-7 \\ 0+21 \end{bmatrix} = \begin{bmatrix} 18 \\ 1 \\ 21 \end{bmatrix}$$

<u>E.x.</u> Compute
$$\begin{bmatrix} 1 & 3 & 1 \\ -3 & 0 & 2 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

E.g. If $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are three column vectors, write the linear combination $\vec{u}_1 - 3\vec{u}_2 + 5\vec{u}_3$ as a matrix product.

$$[\vec{u}_1 \ \ \vec{u}_2 \ \ \vec{u}_3] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}.$$

Matrix multiplication is a concise way of representing linear combinations.

If A is an $m \times n$ matrix, and $\vec{b} \in \mathbb{R}^m$ is a column vector, can we solve the matrix equation

$$A\vec{x} = \vec{b} \quad \text{for} \quad \vec{x} \in \mathbb{R}^n ?$$

E.g. Given $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 6 \\ -3 & 2 & 7 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Solve:

$$A\vec{x} = \vec{b}, \text{ i.e., } \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 6 \\ -3 & 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Consider augmented matrix

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ -4 & 2 & 6 & 0 \\ -3 & 2 & 7 & 1 \end{array}\right] \xrightarrow{\text{row ops}} \text{RREF}$$

to find the solution.

Thm: $A\vec{x} = \vec{b}$ can be solved

iff $\vec{b}$ is a linear combination of the columns of A

iff $\vec{b} \in$ span of the columns of A.

if and only if

E.x. $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 6 \\ -3 & 2 & 7 \end{bmatrix}$

Can I solve $A\vec{x} = \vec{b}$ for any choice of $\vec{b}$?

i.e. Is every vector $\vec{b} \in \mathbb{R}^3$ in the span of the columns of A?

(14)

Thm: Let A be an $m\times n$ matrix.

The following four statements are equivalent.

a) $A\vec{x}=\vec{b}$ can be solved for $\vec{x}$, for any $\vec{b}\in\mathbb{R}^m$.

b) Every vector $\vec{b}\in\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$

d) A has a pivot in each row.

E.g. 1) Let $A=\begin{bmatrix}3&5&-4&1\\-3&-2&4&2\\6&1&-8&-7\end{bmatrix}$. Do the columns of A span $\mathbb{R}^3$?

No!

$$A \xrightarrow{\text{row ops.}} \cdots \longrightarrow \begin{bmatrix}1&0&-4/3&-4/3\\0&1&0&1\\0&0&0&0\end{bmatrix}$$

because last row does not have a pivot.

2) Is $\begin{bmatrix}0\\0\\1\end{bmatrix}$ in the span of the columns of A?

No!

Consider

$$[A\,|\,\vec{b}]=\left[\begin{array}{cccc|c}3&5&-4&1&0\\-3&-2&4&2&0\\6&1&-8&-7&1\end{array}\right]\xrightarrow{\text{row ops.}}\cdots\longrightarrow\left[\begin{array}{cccc|c}\boxed{3}&5&-4&1&0\\0&\boxed{3}&0&3&0\\0&0&0&0&\boxed{1}\end{array}\right]$$

Note:

$$\begin{cases}3x_1+x_2-x_3=1\\-x_1+2x_2+5x_3=0\end{cases}\iff\begin{bmatrix}3&1&-1\\-1&2&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

system of equations — matrix equation

$$x_1\begin{bmatrix}3\\-1\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix}+x_3\begin{bmatrix}-1\\5\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\iff\left[\begin{array}{ccc|c}3&1&-1&1\\-1&2&5&0\end{array}\right]$$

vector eq. — augmented matrix

1.5) Solution sets of Linear systems.

Def: A system of equations is <u>homogeneous</u> if it has the form $A\vec{x} = \vec{0}$.

$\vec{0} \rightarrow$ zero vector $= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Note: There is always at least one solution $\vec{x} = \vec{0}$. But there may be more.

E.g. Solve $\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \end{cases}$.

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

x_3 free.

$$\Rightarrow \begin{cases} x_1 \quad - 4/3\, x_3 = 0 \\ x_2 \quad = 0. \end{cases}$$

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ solution set $= \text{span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Inhomogeneous systems are linear systems that are not homogeneous, i.e., $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$.

E.g. Let's solve

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -3 & 0 \end{array}\right]$$

$\Rightarrow$ x_3 and x_4 are free.

Let $x_3 = s$ and $x_4 = t$.

$\Rightarrow$ Sol:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Sol. set} = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

and compare to

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 4 \\ 1 & 0 & 1 & 2 & 3 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$x_3 = s \quad x_4 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}}$$

Thm: Let A be a matrix, and denote the (parametric) solution set of the homogeneous equation

$$A\vec{x} = \vec{0}$$

as $\vec{v}_h$.

a) Even though $A\vec{x} = \vec{0}$ is always consistent ($\vec{x} = \vec{0}$ is always in $\vec{v}_h$), for given $\vec{b} \neq \vec{0}$, the equation

$$A\vec{x} = \vec{b}$$ may be inconsistent.

b) If $A\vec{x} = \vec{b}$ is consistent (i.e. $\vec{b}$ is in the span of the columns of A) and if $\vec{x} = \vec{p}$ is any particular solution, then the general solution is

$$\vec{x} = \vec{v}_h + \vec{p}.$$

1.7) Linear Independence.

- Def: A collection of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is called <u>linearly dependent</u> if at least one of them is in the span of the others.

  If a set of vectors is not linearly dependent, we call the vectors <u>linearly independent</u>.

<u>E.g.</u> Are these vectors linearly independent?

a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$.

No! $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b) $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$.

The question is equivalent to "does $x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \vec{0}$ have a nonzero solution??"

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 7 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$\Rightarrow x_1 = 0, \ x_2 = 0.$

$\Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ linearly ind.

<u>Thm</u>: vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are <u>linearly independent</u> <u>if and only</u> if the vector equation

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \ldots + x_n\vec{u}_n = \vec{0}$$

has <u>only</u> the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

<u>Cor</u>: The columns of a matrix A are linearly independent if and only if $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

<u>E.g.</u> Are $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}$ linearly independent?

Solve

$$\begin{bmatrix}1 & 2 & 0\\1 & 0 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\1 & 0 & 1 & 0\\0 & 1 & 2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\0 & -2 & 1 & 0\\0 & 1 & 2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\0 & 1 & -1/2 & 0\\0 & 1 & 2 & 0\end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\0 & 1 & -1/2 & 0\\0 & 0 & 5/2 & 0\end{array}\right]$$

$\vec{x} = \vec{0}$. $\Leftarrow$ ∴ unique solution.

The vectors are linearly ind.

- What if one of the vectors is $\vec{0}$?

A: linearly dependent.

E.g.

$$0\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\cdot\begin{bmatrix}2\\3\\4\end{bmatrix} + 48\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

- What if there is only one vector?

$\{\vec{v}_1\}$ is linearly independent if and only if $\vec{v} \neq \vec{0}$.

Thm: The columns of a matrix A are linearly independent if and only if A has a pivot position in every column.
(Otherwise, the first non-pivotal column is a linear combination of the preceding ones, with coefficients read off ~~th~~ of the RREF.)

Cor: If A has more columns than rows, its columns are linearly independent.