1

Math 18: Linear Algebra

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- Office Hours: 9:30 – 10:30AM Monday & Web at APM 6342
- Course Website: www.thanghuynh.io/teaching/math18_winter19/home
  ⇒ . Syllabus
  . Grading info.
  . Homework.
  . Exam schedule.

- Textbook: Linear Algebra and Its Applications, 5th Edition.
  by Lay, Lay, and McDonald.
  Note: . you only need the MyMathLab access code. This comes with an ebook.
  you can use MyMathLab for the first 14 days without paying.
- Piazza.
- Lecture notes posted online.
- Need to know:
  Final: March 18, 11:30am – 2:30pm, Location: TBA.
  Exam 1: Jan 30, in class.
  Exam 2: Feb 27, in class.
  Homework: → MyMathLab Online HW (required).
  ↳ Offline HW (recommended).
  MatLab assignments + quiz.
  Reading, discussion section, lecture.

Linear Algebra is the branch of mathematics concerning linear equations and their representations through matrices and vector spaces.

(1.1) Systems of Linear Eqs:

- Def: A linear eq in variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $\underbrace{a_1, \dots, a_n}$ and b are some (known) numbers

a_i called the coefficients of x_i.

- E.g: 1) $x_1 + 2x_2 + 3x_3 = 0$.
  2) $x_1 + 2x_2 = 3x_1$
  3) Q: is $x_1^2 = x_2$ a linear equation?
     A: No.

- Def: A system of linear eqs (linear system) is a set of linear eqs.

  Two systems are equivalent if they have the same set of solutions.

  E.g. $\begin{cases} x_1 = 2 \\ x_1 + x_2 = x_2 \end{cases}$ and $\begin{cases} x_1 = x_1 \\ x_2 = 1 \\ x_2 = 2 \end{cases}$

  are equivalent (both have no solutions)

- Def: A linear system is called "consistent" if it has a solution. Otherwise, it is inconsistent.

• **Def:** An <u>$m \times n$ matrix</u> is an array of numbers arranged in m rows and n ~~m~~ columns.

E.g. a 2×4 matrix:

2 rows $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

4 columns.

<u>Note:</u> we can encode a system of m equations in n variables as an $m\times(n+1)$ matrix.

E.g. $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 3x_1 + x_2 + 6x_3 = 8 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 1 & 6 & 8 \end{array}\right]$.

This is called the "augmented matrix" of the system.

In general,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\Rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

The $m\times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

is called the <u>matrix of coefficients</u>.

E.x. 1) What's the augmented matrix of

$$x_1 + x_2 - x_3 = 2$$
$$x_2 + 1 = x_3.$$

2) What's the system of equations whose augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{array}\right]$$

Let's solve a linear system, and see what happens to the augmented matrix at each step:

$$2x_1 + 4x_2 = 6 \qquad \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 3 & -1 & 7 \end{array}\right]$$
$$3x_1 - x_2 = 7$$

dividing the first eqn by 2:

$$x_1 + 2x_2 = 3 \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 7 \end{bmatrix}$$
$$3x_1 - x_2 = 7$$

Subtracting 3x (eqn 1) from eqn 2:

$$x_1 + 2x_2 = 3 \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -2 \end{bmatrix}$$
$$0x_1 - 7x_2 = -2$$

~~Subtracting 2x (eqn 2) from eqn 1~~:

$\cancel{x_1 +}$

Multiplying eqn 2 by 1/-7:

$$x_1 + 2x_2 = 3 \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2/7 \end{bmatrix}$$
$$x_2 = 2/7$$

Subtracting 2x (eqn 2) from eqn 1:

$$x_1 + 0x_2 = 17/7 \qquad \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \end{bmatrix}$$
$$x_2 = 2/7$$

* <u>Idea</u>: We can replace a linear system with an equivalent and simpler one by applying the following three operations to the augmented matrix:

- Multiplying a row by a number.
- Add a multiple of one row to another.
- Reorder the rows.

These are called the <u>elementary row operations</u>.

<u>Def</u>: Two matrices are <u>row equivalent</u> if you can get from one to the other by doing elementary <u>row operations</u>.

<u>Ex.</u> Solve

$$x_2 + x_3 = -1$$
$$x_1 + x_2 = 1$$
$$2x_1 - x_3 + x_2 = 3$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right] \xrightarrow[R_3 - 2R_2]{} \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - R_2]{} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 - x_3 = 2 \\ x_2 + x_3 = -1 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2 + x_3 \\ x_2 = -1 - x_3 \end{cases}$$

Solution: $(x_1, x_2, x_3) = (2+t, -1-t, t)$ where t is any number. (The system is consistent, but there are infinitely many solutions)

What does "simpler" mean for augmented matrix?

Def: A matrix is in row-echelon form if

1) All rows of zeros are on the bottom.

2) The left most non-zero entry of each row (the leading entry) is to the right of the leading entry of the row above it.

E.g.
$$\left[\begin{array}{ccccc|c} 0 & \boxed{2} & 1 & \pi & 0 & 4 \\ 0 & 0 & 0 & \boxed{3} & 2 & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{"leading entry"} \quad \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

Not row-echelon:
$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 3 \\ 0 & \boxed{4} & 5 \\ 0 & \boxed{6} & 7 \end{array}\right] \qquad \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 1 & 1 \\ \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

In general,

- Row Echelon Form:
$$\left[\begin{array}{ccccc|c} 0 & \blacksquare & \times & \times & \times & \times \\ 0 & 0 & \blacksquare & \times & \times & \times \\ 0 & 0 & 0 & \blacksquare & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \end{array}\right]$$

- Reduced Row Echelon Form:
$$\left[\begin{array}{cccccc|c} 0 & \boxed{1} & \times & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \boxed{1} & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \boxed{1} & \times & \times \end{array}\right]$$

Thm: By performing row operations, every matrix can be transformed to reduced row echelon form, and that form is unique.

E.g. Find RREF of the following augmented matrix.

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\xrightarrow[\cancel{\frac{2}{3}R_2 - R_2} \quad R_2 \to R_2/2]{R_1 \to R_1/3} \left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & +3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_2} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & 3 & 5 & -4 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{R_1 - 5R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & 3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right]$$

pivot columns

→ the original lin. system is equivalent to

$$\begin{aligned} \boxed{x_1} \quad - 2x_3 + 3x_4 &= -24 \\ \boxed{x_2} \quad - 2x_3 + 2x_4 &= -7 \\ \boxed{x_5} &= 4. \end{aligned}$$

pivot variables

free variables

Ex: Given $2x_1 - x_2 = h$
$-6x_1 + 3x_2 = k$

a) For which h and k is the system consistent?

b) For which h and k is there a unique solution?

Let's consider the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & h \\ -6 & 3 & k \end{array}\right] \xrightarrow{R_2 + 3R_1} \left[\begin{array}{cc|c} 2 & -1 & h \\ 0 & 0 & k+3h \end{array}\right].$$

- If $k + 3h = 0$, then the system is consistent and has infinitely many solutions.
- If $k + 3h \neq 0$, then the system is inconsistent.

* Existence & Uniqueness:

- If the right most column of the augmented matrix is pivotal, the system is inconsistent.

E.g.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right] \\ \uparrow \quad\quad\quad \uparrow \quad \uparrow \\ \text{pivot columns} \end{array}$$

this system is inconsistent.

- Otherwise, the system is consistent.
  - If there are free variables, there are ∞ many sols.
  - If not (i.e. every colum, but not the last one, is pivotal), the system has unique sol.

1.3) Vector Equations.

<u>Def:</u> A <u>vector</u> (or column vector) is a list of real numbers in a column.

<u>E.g.</u> $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ both are in $\mathbb{R}^2$.

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^4.$$

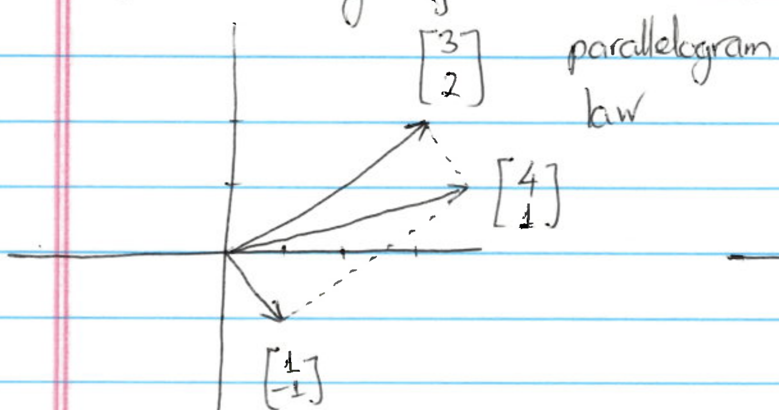
- <u>Addition:</u> column vectors can be added.

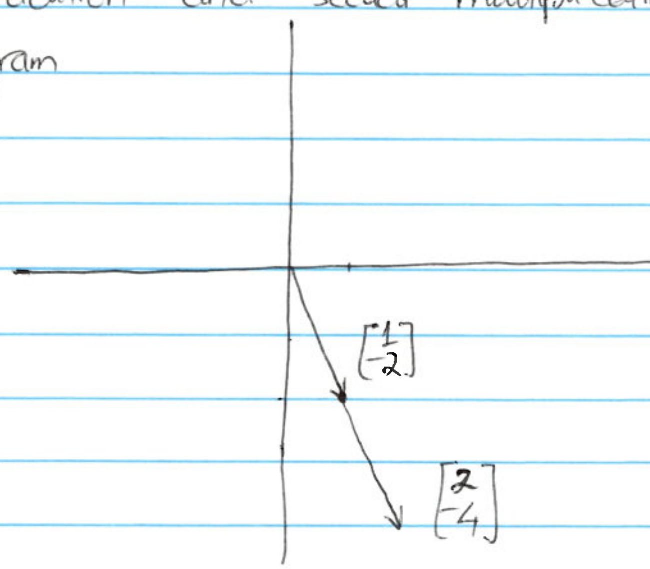
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- <u>Multiplication by scalars:</u> column vectors can be multiplied by scalars.

$$2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

- Geometry of vector addition and scalar multiplication





Vector arithmetic works just like real number arithmetic.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$$

$$\text{where } \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}.$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}.$$

$$(c+d)\vec{u} = c\vec{u} + d\vec{u}.$$

$$c(d\vec{u}) = (cd)\vec{u}.$$

$$1\vec{u} = \vec{u}.$$

* Linear Combinations:

Given vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, a linear combination of these vector is a vector of the form

$$\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_n \vec{u}_n$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

E.g. $$2\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.$$

- Span: The span of a collection of vectors is the set of all linear combinations of those vectors.

(11)

E.g. Is $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ in the span generated by $\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}$??

Yes! (what are α_1, α_2?)

i.e. can we express $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ in the form $\alpha_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$?

i.e. Find scalars α_1, α_2 such that

$$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -2\alpha_1 + 5\alpha_2 \\ -5\alpha_1 + 6\alpha_2 \end{bmatrix}.$$

$\Rightarrow$ Solve the lin. system

$$\begin{cases} \alpha_1 + 2\alpha_2 = 7 \\ -2\alpha_1 + 5\alpha_2 = 4 \\ -5\alpha_1 + 6\alpha_2 = -3 \end{cases} \Rightarrow \left[\begin{array}{cc:c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$$

$$\xrightarrow[R_3+5R_1]{R_2+2R_1} \left[\begin{array}{cc:c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ consistent.

* The question: "Is $\vec{w}$ in the span of $\{\vec{v}_1, \dots, \vec{v}_n\}$?"
is the same as the question
"Is the system whose augmented matrix is $[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n \mid \vec{w}]$ consistent?"

$\Rightarrow$ To answer this question, we use row echelon form and check to see if the augmented column is pivotal!