HW05

Problem 1. Let's start with 2×2 matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 2 & 1\cdot 3 + 2\cdot 1 \\ 3\cdot 4 + 4\cdot 2 & 3\cdot 3 + 4\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 4 & 1\cdot 3 \\ 3\cdot 4 & 3\cdot 3 \end{bmatrix} + \begin{bmatrix} 2\cdot 2 & 2\cdot 1 \\ 4\cdot 2 & 4\cdot 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix}.$$

For general matrices,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}\cdot b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{11} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}\cdot b_{21} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1p} \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1p} \\ \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1p} \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}b_{n1} & a_{1n}b_{n2} & \cdots & a_{1n}b_{np} \\ a_{2n}b_{n1} & a_{2n}b_{n2} & \cdots & a_{2n}b_{np} \\ \vdots & \vdots & & \vdots \\ a_{mn}b_{n1} & a_{mn}b_{n2} & \cdots & a_{mn}b_{np} \end{bmatrix}$$

$$= A(:,1)B(1,:) + \cdots + A(:,n)B(n,:)$$

Problem 2.

$$A_{2\times 3} B_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}*b_{11} + a_{12}*b_{21} + a_{13}*b_{31} & a_{11}*b_{12}+a_{12}*b_{22}+a_{13}*b_{32} & a_{11}*b_{13}+a_{12}*b_{23}+a_{13}*b_{33} \\ a_{21}*b_{11} + a_{22}*b_{21} + a_{23}*b_{31} & a_{21}*b_{12}+ a_{22}*b_{22} +a_{23}*b_{32} & a_{21}*b_{13}+a_{22}*b_{23}+a_{23}*b_{33} \end{bmatrix}$$

there are 18 (*) operations $\Big\}$ ⇒ total operations
and 12 (+) operations. $= 30$.

Note that, in class we learned that the number of operations we need to multiply two matrices $A_{m\times n}$ $B_{n\times p}$ is $O(mnp)$. ⟵ This is only when $m, n,$ and p are very large.

Problem 3.

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 11 \\ 1 & -7 & -10 \end{bmatrix}$$

a) $\mathbb{E}[X] = \frac{1}{3}\cdot 3\, A(:,1)B(1,:) + \frac{1}{3}\cdot 3A(:,2)B(2,:) + \frac{1}{3}\cdot 3A(:,3)B(3,:)$

$= A(:,1)B(1,:) + A(:,2)B(2,:) + A(:,3)B(3,:)$

$= AB.$

$\mathrm{Var}[X] = \sum_{k=1}^{3} 3\,\|A(:,k)\|_2^2 \,\|B(k,:)\|_2^2 - \|AB\|_F^2$

$= \sum_{k=1}^{3} 3(2)(30) + 3(4)(26) + 3(5)(\overset{82}{\cancel{110}}) - 308.$

$= \cancel{1834}\ 1414.$

b) $\|A\|_F^2\, \|B\|_F^2 = (1+1+1+2^2+2^2)(1+5^2+2^2+1+5^2+1^2+9^2)$
$= \cancel{7}\,(11)(138) = 1518.$

b) $Var[X] = 1414 < \|A\|_F^2 \|B\|_F^2 = 1518.$

b) In class, we take $P_k = \dfrac{\|A(:,k)\|^2}{\|A\|_F^2}$ (length squared sampling technique).

then we get $Var[X] \leq \|A\|_F^2 \|B\|_F^2$.

But in this exercise, $P_k \geq \dfrac{1}{3}$.

and we obtain

$Var[X] \geq \|A\|_F^2 \|B\|_F^2$.

Problem 4.

$A = \begin{bmatrix} 1 & -5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & -9 \end{bmatrix}.$

a) $|A|_1 = |1| + |-5| + |2| + |1| + |5| + |1| + |-9|.$

$= 24.$

Note that $|A|_1$ is not $\|A\|_1$ which is $\|A\|_1 = 10.$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix}.$

c) $P_{11} = \dfrac{1}{24}, \quad P_{12} = \dfrac{5}{24}, \quad P_{13} = \dfrac{2}{24},$

$P_{21} = \dfrac{1}{24}, \quad P_{22} = 0, \quad P_{23} = \dfrac{5}{24}$

$P_{31} = 0, \quad P_{32} = \dfrac{1}{24}, \quad P_{33} = \dfrac{9}{24}.$

d) See lecture note.

e) $Y = X - A \Rightarrow E[Y] = E[X - A] = E[X] - A = 0.$

$\|Y\|_2 = \|X - A\|_2 \leq \|X\|_2 + \|A\|_2 \leq 24 + \|A\|_2.$

$\|X\|_2 = \| \dfrac{1}{P_{ij}} A_{i,j} \|_2 = 24 \| \dfrac{A_{ij}}{|a_{ij}|} \|_2 \leq 24$