HW05

DUE WEEK 9 AND 10

Reading: Lecture notes and Chapter 6, *Foundations of Data Science* by Avrim Blum, John Hopcroft, and Ravindran Kannan

Problem 1. Let A and B be $m \times n$ and $n \times p$ matrices, respectively. Let $A(:,k)$ be the kth column of A, and $B(k,:)$ the kth row of B. Show that

$$AB = \sum_{k=1}^{n} A(:,k)B(k,:).$$

Problem 2. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. How many operations do we need to calculate AB? What are these operations?

Problem 3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 9 \end{bmatrix}$. Define a matrix-valued random variable X by

$$X = 3A(:,k)B(k,:) \text{ with probability } 1/3,$$

where $A(:,k)$ is the kth column of A, and $B(k,:)$ the kth row of B.

a) Calculate $\mathbb{E}[X]$ and $\text{Var}[X] = \mathbb{E}[\|X - AB\|_F^2]$.

b) Show that $\text{Var}[X] \leq \|A\|_F^2 \|B\|_F^2$.

Problem 4. Let $A = \begin{bmatrix} 1 & -5 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & -9 \end{bmatrix}$.

a) Let $|A|_1 = \sum_{i,j} |a_{i,j}|$. Find $|A|_1$.

b) Let $A_{i,j}$ be the 3×3 matrix whose entries are all zeros except entry (i,j) which is set to $a_{i,j}$. Write A in term of the matrices $A_{i,j}$.

c) Let $p_{i,j} = \frac{|a_{i,j}|}{|A|_1}$. Find $p_{i,j}$ for all i and j.

d) Define a matrix-valued random variable X by $X = \frac{1}{p_{i,j}} A_{i,j}$ with probability $p_{i,j}$. Show that $\mathbb{E}[X] = A$.

e) Let $Y = X - A$. Show that $\mathbb{E}[Y] = 0$ and $\|Y\|_2 \leq 24 + \|A\|_2$.