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HW4.

Exercise 3.5.

$$a) \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow M^T M = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}.$$

i) Find eigenvalues of $M^T M$:

$$\det \begin{bmatrix} 10-\lambda & 1 \\ 1 & 10-\lambda \end{bmatrix} = (10-\lambda)^2 - 1 = 0.$$

$$(9-\lambda)(11-\lambda) = 0.$$

$$\lambda_2 = 9 \quad \text{or} \quad \lambda_1 = 11.$$

ii) Find eigenvectors:

$$\text{For } \lambda_1 = 11, \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\text{For } \lambda_2 = 9: \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

iii) Since $\lambda_1 = 11$, and $\lambda_2 = 9$, $\sigma_1 = \sqrt{11}$ and $\sigma_2 = 3$.

$$M v_1 = \sigma_1 u_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_1 u_1.$$

$$\Rightarrow \sigma_1 u_1 = \begin{bmatrix} 2/\sqrt{2} \\ 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{11}} \begin{bmatrix} 2/\sqrt{2} \\ 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{22}} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$M v_2 = \sigma_2 u_2 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_2 u_2.$$

$$\Rightarrow u_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 3/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\Rightarrow M = \begin{bmatrix} 2/\sqrt{22} & 0 \\ 3/\sqrt{22} & 1/\sqrt{2} \\ 3/\sqrt{22} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{11} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$b) \quad M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow M^T M = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix}$$

• Find eigenvalues:

$$\det \begin{bmatrix} 14-\lambda & 6 \\ 6 & 14-\lambda \end{bmatrix} = (14-\lambda)(14-\lambda) - 6^2 = 0.$$

$$\lambda^2 - 28\lambda + 140 - 36 = 0.$$

$$(20-\lambda)(8-\lambda) = 0$$

$$\lambda^2 - 28\lambda + 104 = 0$$

$$\lambda_1 = 20 \quad \text{and} \quad \lambda_2 = 8$$

• Find eigenvectors:

$$\text{For } \lambda_1 = 20, \quad \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\text{For } \lambda_2 = 8, \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\bullet \quad M v_1 = \sigma_1 u_1 \Rightarrow \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \sqrt{20} u_1 \Rightarrow u_1 = \begin{bmatrix} 2/\sqrt{40} \\ 2/\sqrt{40} \\ 4/\sqrt{40} \\ 4/\sqrt{40} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$$

$$\bullet \quad M v_2 = \sigma_2 u_2 \Rightarrow \begin{bmatrix} 2/\sqrt{2} \\ -2/\sqrt{2} \\ 2/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix} = \sqrt{8} u_2 \Rightarrow u_2 = \begin{bmatrix} 2/\sqrt{16} \\ -2/\sqrt{16} \\ 2/\sqrt{16} \\ -2/\sqrt{16} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1/\sqrt{10} & 1/2 \\ 1/\sqrt{10} & -1/2 \\ 2/\sqrt{10} & 1/2 \\ 2/\sqrt{10} & -1/2 \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$3) \quad A = U \Sigma V^T = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

U : customer - concept (e.g. cuisine) similarity matrix.

V : restaurant - concept (e.g. cuisine) similarity matrix.

$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ^{1st cuisine (e.g. Western)} restaurant 1.
restaurant 2

$\Rightarrow$ restaurant 2 is likely a Western one, but restaurant 1

may be not.
2nd cuisine (e.g. Asian)

$u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ restaurant 1.
restaurant 2.

$\Rightarrow$ restaurant 1 is likely an Asian one, but restaurant 2 is not.

For u_1 , it tells us how the customers like the 1st cuisine.
 σ_i 's are the strength of the concepts (e.g. Western and Asian cuisine).

$\sigma_1 - \sigma_2 = 4 - 2 = 2 \Rightarrow$ the 1st concept is stronger than the second one.

Exercise 3.6. $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix}$

$$1) B = A^T A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$$

$$x^{(1)} = Bx / \|Bx\|_2 = \begin{bmatrix} 4 \\ 16 \end{bmatrix} \cdot \frac{1}{\sqrt{4^2 + 16^2}} = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x^{(2)} = Bx^{(1)} / \|Bx^{(1)}\|_2 = \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 64 \end{bmatrix} / \frac{4}{\sqrt{17}} \sqrt{257} = \frac{1}{\sqrt{257}} \begin{bmatrix} 1 \\ 16 \end{bmatrix}$$

$$x^{(3)} = Bx^{(2)} / \|Bx^{(2)}\|_2 = \frac{1}{\sqrt{257}} \begin{bmatrix} 4 \\ 256 \end{bmatrix} / \frac{1}{\sqrt{257}} \sqrt{4^2 + 256^2}$$

$$\Rightarrow x^{(3)} = \frac{1}{\sqrt{65552}} \begin{bmatrix} 4 \\ 256 \end{bmatrix} \text{ is an estimate of } v_1.$$

2) $\lambda_1 = 16$ and $\lambda_2 = 4$ are eigenvalues of B .

$$\Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Av_1 = \sigma_1 v_1 \Rightarrow \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = 4v_1 \Rightarrow v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$Av_2 = \sigma_2 v_2 \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 2v_2 \Rightarrow v_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Note: the first component of $x^{(3)}$ is small. If we iterate more, the first component of that vector is getting smaller $\Rightarrow$ closer to $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Exercise 3.10: Verify that

$$\sum_{i=1}^r c_i \vec{x}_i \vec{y}_i^T = \underbrace{\begin{bmatrix} \vec{x}_1 & \dots & \vec{x}_r \end{bmatrix}}_X \underbrace{\begin{bmatrix} c_1 & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & c_r \end{bmatrix}}_C \underbrace{\begin{bmatrix} \vec{y}_1^T \\ \vdots \\ \vec{y}_r^T \end{bmatrix}}_{Y^T}.$$

Recall that Matrices A and B are identical if and only if for all vectors v , $Av = Bv$ (see Lemma 3.3). We will show that for any vector v ,

$$\left(\sum c_i x_i y_i^T \right) v = X C Y^T v.$$

$$\begin{aligned} 1) \text{ LHS} &= \sum_{i=1}^r c_i x_i y_i^T v = \sum_{i=1}^r c_i x_i \langle y_i, v \rangle \\ &= \sum_{i=1}^r \underbrace{c_i \langle y_i, v \rangle}_{\text{scalar}} x_i \end{aligned}$$

$$= \begin{bmatrix} x_1 & x_2 & \dots & x_r \end{bmatrix} \begin{bmatrix} c_1 \langle y_1, v \rangle \\ c_2 \langle y_2, v \rangle \\ \vdots \\ c_r \langle y_r, v \rangle \end{bmatrix}.$$

$$2) \text{ RHS} = X C Y^T v = X C \begin{bmatrix} \langle y_1, v \rangle \\ \vdots \\ \langle y_r, v \rangle \end{bmatrix} = X \begin{bmatrix} c_1 \langle y_1, v \rangle \\ \vdots \\ c_r \langle y_r, v \rangle \end{bmatrix}.$$

$$\Rightarrow \text{ LHS} = X \begin{bmatrix} c_1 \langle y_1, v \rangle \\ \vdots \\ c_r \langle y_r, v \rangle \end{bmatrix} = \text{RHS}.$$

Exercise 3.12.

1) Let $u_1, \dots, u_r$ be the left singular vectors of A . Then

$$A = U \Sigma V^T.$$

$$\Rightarrow \bar{A}^T A = (U \Sigma V^T)^T A U \Sigma V^T = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T.$$

$$= V \Sigma \Sigma V^T.$$

$$= V \Sigma^2 V^T.$$

$$= \sum_{i=1}^r \sigma_i^2 v_i v_i^T.$$

2) For any $j \in \{1, \dots, r\}$:

$$\begin{aligned} \bar{A}^T A v_j &= \left(\sum_{i=1}^r \sigma_i^2 v_i v_i^T \right) v_j \\ &= \sum_{i=1}^r \sigma_i^2 v_i \langle v_i, v_j \rangle. \end{aligned}$$

$$\Rightarrow \bar{A}^T A v_j = \sigma_j^2 v_j \quad \text{since } \langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\Rightarrow v_j$ is an eigenvector of $\bar{A}^T A$.

3) ~~Test~~ (Challenging see Trefethen & Bau textbook)

I will not ask this question on quiz & exam.

Exercise 3.13:

1) $\|A_k\|_F^2 = \sum_{i=1}^k \sigma_i^2$ (See Lemma 3.2)

2) $\|A_k\|_2^2 = \sigma_1^2$.

Since A_k has the largest singular value is σ_1 .

3) $A - A_k = \sum_{i=k+1}^r \sigma_i u_i v_i^T$

$\Rightarrow \|A - A_k\|_F^2 = \sum_{i=k+1}^r \sigma_i^2$

4) $\|A - A_k\|_2^2 = \sigma_{k+1}^2$ since the largest singular value of $A - A_k$ is σ_{k+1} .

Exercise 3.16: (None) I was not able to go over all aspects of the power method!

Problem 1: $M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$

a) $B = M^T M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

Find eigenvalues and eigenvectors of B .

$$\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)^2 - 1 = 0.$$

$$(6-\lambda)(4-\lambda) = 0.$$

$$\lambda_1 = 6 \quad \text{and} \quad \lambda_2 = 4.$$

For $\lambda_1 = 6$, solve

$$(B - 6I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} = 0.$$

$$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 4$,

$$(B - 4I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v} = 0 \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find $\vec{u}_i$'s:

$$\text{So } \vec{u}_1 = \frac{1}{\sigma_1} M \vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{u}_2 = \frac{1}{\sigma_2} M \vec{v}_2 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\therefore M = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$b) B = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}.$$

$$\vec{x}^{(1)} = B\vec{x} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\vec{x}^{(2)} = B\vec{x}^{(1)} = 6 \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\vec{x}^{(3)} = B\vec{x}^{(2)} = 6^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{normalize } \vec{x}^{(3)} \Rightarrow \frac{\vec{x}^{(3)}}{\|\vec{x}^{(3)}\|_2} = \frac{6^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{6^6 + 6^6}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 2:

$$a) A_1 = \sqrt{3} \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

is the best rank-1 approximation matrix to A .

b) Since $\sigma_1, \sigma_2, \sigma_3$ dominate all other singular values, take

$$M_3 = \sigma_1 \vec{u}_1 \vec{u}_1^T + \sigma_2 \vec{u}_2 \vec{u}_2^T + \sigma_3 \vec{u}_3 \vec{u}_3^T.$$