Math 152 - HW02.

1) Let $X_i = \begin{cases} 1 & \text{if Alice wins, with probability } 0.7 \\ 0 & \text{if Alice loses, with probability } 0.3 \end{cases}$.

(X_i is the random variable indicating whether Alice wins or not in ~~the~~ the ith game).

Let $X = X_1 + X_2 + \dots + X_n$. Thus, X is the random variable indicating the number of games Alice wins in the tournament.

Then, $\mathbb{E}[X_i] = 0.7$ and $\mu = \mathbb{E}[X] = n(0.7)$.

~~App~~ Recall that Alice loses ~~looses~~ ~~loose~~ the tournament if she wins less than half of the games, i.e., $X \leq \frac{n-1}{2}$.

By Chernoff's bound for $\{0,1\}$-valued r.v's. (See other ~~oth~~ useful forms in the lecture note):

$$P(X \leq (1-\varepsilon)\mu) \leq e^{-\mu\varepsilon^2/2}.$$

Set $(1-\varepsilon)\mu = \frac{n-1}{2}$, and find ε,

$$0.7n(1-\varepsilon) = 0.5n - 0.5$$

$$0.7n - \varepsilon 0.7n = 0.5n - 0.5$$

$$0.2n + 0.5 = \varepsilon\, 0.7n.$$

$$\frac{2}{7} + \frac{5}{7n} = \varepsilon.$$

$$\Rightarrow \quad \varepsilon = \frac{2}{7} + \frac{5}{7n} > \frac{2}{7}. \Rightarrow \varepsilon^2 > \frac{4}{49}.$$

$$\therefore \quad P\left(X \leq \frac{n-1}{2}\right) = P(X \leq (1-\varepsilon)\mu) \leq e^{-\mu\varepsilon^2/2} < e^{-0.7n\left(\frac{4}{49}\right)/2}.$$

$$\therefore \quad P\left(X \leq \frac{n-1}{2}\right) \leq e^{-n/5}.$$

2) Let X be the number of times that a 6 occurs over n throws of the die.

Let $p = \mathbb{P}(X \geq \frac{n}{4})$.

a) Markov's inequality:

$$p = \mathbb{P}(X \geq \tfrac{n}{4}) \leq \frac{\mathbb{E}[X]}{\frac{n}{4}} = \frac{4\mathbb{E}[X]}{n}.$$

We need to evaluate $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \cancel{n}\sum_{i=1}^{n} \mathbb{E}[X_i] = \frac{n}{6}.$$

where X_i is the result we get at the ith throws, i.e. $X_i = \begin{cases} 1 & \text{w/ prob } 1/6 \\ 0 & \text{otherwise w/ prob } 5/6 \end{cases}$

$$\therefore \boxed{p \leq \frac{4\left(\frac{n}{6}\right)}{n} = \frac{4}{6} = \frac{2}{3}}.$$

b) Chebyshev's inequality:

$$\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \frac{\mathrm{Var}[X]}{t^2}$$

$$\mathrm{Var}[X] = \mathrm{Var}\left[\sum_i X_i\right] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = n \cdot \frac{5}{36} = \frac{5n}{36}.$$

$$\therefore \quad \mathbb{P}(X \geq \tfrac{n}{4}) = \mathbb{P}\left(X - \tfrac{n}{6} \geq \tfrac{n}{4} - \tfrac{n}{6}\right)$$

$$= \mathbb{P}\left(X - \mathbb{E}[X] \geq \tfrac{n}{12}\right)$$

$$\leq \mathbb{P}\left(|X - \mathbb{E}[X]| \geq \tfrac{n}{12}\right)$$

Chebyshev's ineq. with $t = \frac{n}{12}$

$$\leq \frac{\mathrm{Var}[X]}{(n/12)^2}$$

$$= \frac{12^2}{n^2} \cdot \frac{5n}{36}$$

$$= \frac{\cancel{1}20}{n}.$$

$$\therefore \boxed{p \leq \frac{20}{n}}$$

c) Chernoff's inequality:

$$\cancel{p = \mathbb{P}\left(X \geq \frac{n}{4}\right)}$$

$$\mathbb{P}\left(X \geq (1+\varepsilon)\mu\right) \leq e^{-\mu\varepsilon^2/4}.$$

We want $(1+\epsilon)\mu = \frac{n}{4} \Rightarrow (1+\epsilon)\frac{n}{6} = \frac{n}{4}$

$$1+\varepsilon = \frac{3}{2} \Rightarrow \varepsilon = \frac{1}{2}.$$

$$\therefore\ p = \mathbb{P}\left(X \geq \frac{n}{4}\right) = \mathbb{P}\left(X \geq (1+\varepsilon)\mu\right) \leq e^{-\frac{n}{6}\cdot\left(\frac{1}{2}\right)^2/4} = e^{-n/96}.$$

$$\therefore\ \boxed{p \leq e^{-n/96}}$$

3) $X_i = \begin{cases} 1 & w/p\ 1/2 \\ -1 & w/p\ 1/2 \end{cases}$

Let $S = \sum_{i=1}^{n} X_i$.

a) Since $Y = |S|$, Y is a non-negative random variable, Markov's inequality holds for for Y. That is, for $t > 0$,

$$\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}.$$

b) Chebyshev's inequality: for S

$$\mathbb{P}\left(|S - \mathbb{E}[S]| \geq t\right) \leq \frac{\mathrm{Var}[S]}{t^2}.$$

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}[X_i] = 0.$$

$$\mathrm{Var}[S] = \mathrm{Var}\left[\sum_i X_i\right] = \sum_i \mathrm{Var}[X_i] = n$$

$$\therefore\ \mathbb{P}(|S| \geq t) \leq \frac{n}{t^2}.$$

c) (This is similar to the way we prove Chernoff's bound).

$$\mathbb{P}(S \geq a) = \mathbb{P}(X_1 + \dots + X_n \geq a)$$

$$= \mathbb{P}(\lambda \sum_i X_i \geq \lambda a) \quad \text{for } \lambda \geq 0$$

$$= \mathbb{P}(e^{\lambda \sum_i X_i} \geq e^{\lambda a}) \quad \text{since } e^x \text{ is monotone.}$$

$$\text{Markov's} \leftarrow \leq e^{-\lambda a} \, \mathbb{E}[e^{\lambda \sum X_i}]$$

Now, let's consider $\mathbb{E}[e^{\lambda \sum_i X_i}]$.

$$\mathbb{E}[e^{\lambda \sum_i X_i}] = \mathbb{E}[e^{\lambda X_1} \cdot e^{\lambda X_2} \cdots e^{\lambda X_n}].$$

$$= \prod_i \mathbb{E}[e^{\lambda X_i}] \quad \text{since } X_i\text{'s are independent.}$$

$$= \prod_i \left(e^{\lambda} \cdot \mathbb{P}(X_i = 1) + e^{-\lambda} \, \mathbb{P}(X_i = -1) \right).$$

$$= \prod_i \frac{e^{\lambda} + e^{-\lambda}}{2}.$$

$$= \prod_i \cosh(\lambda)$$

$$= (\cosh(\lambda))^n.$$

Consider $\lambda \in [0, 1]$, then $\cosh(\lambda) \leq e^{\lambda^2/2}$.

$$\therefore (\cosh(\lambda))^n \leq e^{\lambda^2 n/2}$$

$$\therefore \mathbb{P}(S \geq a) \leq e^{-\lambda a} \, \mathbb{E}[e^{\lambda \sum_i X_i}] = e^{\frac{\lambda^2 n}{2} - \lambda a}$$

Recall that min of $\frac{\lambda^2 n}{2} - \lambda a$ occurs at $\lambda = \cancel{\frac{2a}{n}}$ $\frac{a}{n}$ ← take this one.

$$\therefore \mathbb{P}(S \geq a) \leq e^{\frac{a^2}{n^2} \cdot \frac{n}{2} - \frac{a^2}{n}} = e^{-a^2/2n}.$$