Math 152 – HW01 – Solution.

1) a) Let $M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}$. Then

$$MM^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix}.$$

and $$M^TM = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$$

b) We need to show that $A^TA = (A^TA)^T$ and $AA^T = (AA^T)^T$. These are based on properties of matrix transposes. (i.e., $(BC)^T = C^TB^T$ and $(B^T)^T = B$).

$$\Rightarrow (A^TA)^T = A^T(A^T)^T = A^TA.$$

and $$(AA^T)^T = (A^T)^T A^T = AA^T.$$

2) Observe that

$$d_1 = c_1 f_1(1) + c_2 f_2(1) + \dots + c_8 f_8(1)$$
$$d_2 = c_1 f_1(2) + c_2 f_2(2) + \dots + c_8 f_8(2)$$
$$\vdots$$
$$d_8 = c_1 f_1(8) + c_2 f_2(8) + \dots + c_8 f_8(8)$$

$$\Rightarrow \vec{d} = F\vec{c}, \text{ where}$$

②

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix} \qquad F = \cancel{\begin{matrix} f_1(1) & f_1(2) & \cdots & f_1(8) \\ f_1(2) & f_2(2) & \cdots & f_2(8) \end{matrix}}$$

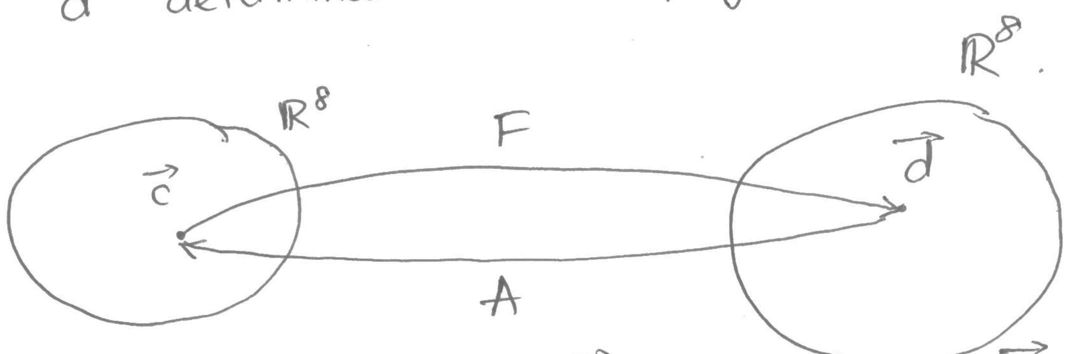
$$F = \begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & f_2(2) & \cdots & f_8(2) \\ \vdots & \vdots & & \vdots \\ f_1(8) & f_2(8) & \cdots & f_8(8) \end{bmatrix} \quad \text{and} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}.$$

The problem says that given any $\vec{d} \in \mathbb{R}^8$, one can find $\vec{c} \in \mathbb{R}^8$ such that $F\vec{c} = \vec{d}$. That is, F is of full rank, or $C(F) = \mathbb{R}^8$.

$\Rightarrow$ F is invertible.

$\Rightarrow$ $\vec{d}$ determines $\vec{c}$ uniquely.

b)



We know that ~~$F\vec{c} = \vec{d}$~~ given any $\vec{d} \in \mathbb{R}^8$, there exists $\vec{c} \in \mathbb{R}^8$ such that $\vec{d} = F\vec{c}$

$\Rightarrow$ $\vec{c} = F^{-1}\vec{d}$.

Hence, $F^{-1} = A$ and $A^{-1} = F$.

$\Rightarrow$ $(A^{-1})_{ij} = F_{ij} = f_j(i)$.

③.

3) Let $R \in \mathbb{R}^{m \times m}$ be a nonsingular upper-triangular matrix. Show that R^{-1} is also upper-triangular.

Since R is invertible, there exists a matrix $A \in \mathbb{R}^{m \times m}$ such that $AR = I_{m \times m}$. (That is, $A = R^{-1}$.)

Let $\vec{r}_j$ be the jth column of R and $\vec{a}_j$ be the jth column of A. Then

$$(*) \quad \begin{cases} \vec{a}_1 r_{11} = \vec{e}_1 \\ \vec{a}_1 r_{12} + \vec{a}_2 r_{22} = \vec{e}_2 . \\ \cdots \\ \vec{a}_1 r_{1j} + \vec{a}_2 r_{2j} + \cdots + \vec{a}_j r_{jj} = \vec{e}_j . \\ \cdots \\ \vec{a}_1 r_{1m} + \vec{a}_2 r_{2m} + \cdots + \vec{a}_m r_{mm} = \vec{e}_m . \end{cases}$$

where $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m\}$ is the standard basis of $\mathbb{R}^m$.

Solving the system $(*)$, we obtain

$$\vec{a}_1 = \vec{e}_1 / r_{11} .$$

$$\vec{a}_2 = (\vec{e}_2 - \vec{a}_1 r_{12}) / r_{22} .$$

$$\vdots$$

$$\vec{a}_j = \left(\vec{e}_j - \sum_{k=1}^{j-1} \vec{a}_k r_{kj}\right) / r_{jj}$$

for $j = 1, \dots, m$.

So, we see that for each column vector $\vec{a}_j$, it has zeros on the components that have indexes larger than j,

$\Rightarrow$ A is an upper triangular matrix.

④.

4) Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$.

Show that A has full rank <u>if and only if</u> given any $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{y}$, then $A\vec{x} \neq A\vec{y}$.

<u>Proof.</u>

($\Rightarrow$) Suppose A is of full rank. Then

$$\text{Null}(A) = \{0\}.$$

Then take any vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ such that $\vec{x} \neq \vec{y}$, i.e., $\vec{x} - \vec{y} \neq 0$.

$$A(\vec{x} - \vec{y}) \neq 0 \quad \text{as} \quad \vec{x} - \vec{y} \notin \text{Null}(A).$$

$$\Rightarrow \quad A\vec{x} \neq A\vec{y}.$$

($\Leftarrow$) Take any $\vec{x} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{0}$. Then

$$A\vec{x} \neq A\vec{0}.$$

$$\Rightarrow \quad A\vec{x} \neq \vec{0}.$$

$$\Rightarrow \quad \vec{x} \notin \text{Null}(A).$$

Therefore, $\text{Null}(A) = \{0\}$.

$\therefore$ A is of full rank.

①

2) a) $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

b) Suppose that Q is an orthogonal matrix, i.e., $Q^T = Q^{-1}$. Then

$$\begin{aligned}\|Qx\|_2^2 &= \langle Qx, Qx \rangle. \\ &= (Qx)^T Qx. \\ &= x^T \underbrace{Q^T Q}\, x. \\ &= x^T\, I\, x. \\ &= x^T x. \\ &= \|x\|_2^2.\end{aligned}$$

$$\Rightarrow \quad \|Qx\|_2 = \|x\|_2.$$

c) Since Q is an orthogonal matrix (why?)

$$\|Qx\|_2 = \|x\|_2 = \sqrt{1^2 + 2^2 + 2^2 + (-1)^2} = \sqrt{10}.$$

3) For $\vec{u} \neq 0, \vec{v} \neq 0 \in \mathbb{R}^m$, let $A = I + uv^T$.

Suppose that A is nonsingular.

Consider $B = I + \alpha uv^T$ for some scalar α, Then

$$\begin{aligned}BA &= (I + \alpha uv^T)(I + uv^T) \\ &= I + uv^T + \alpha uv^T + \alpha u(v^T u)v^T. \\ &= I + \left(1 + \alpha + \alpha\langle v, u\rangle\right) uv^T.\end{aligned}$$

- If $\langle v, u\rangle \neq -1$, then take $\alpha = \dfrac{-1}{1 + \langle v, u\rangle}$

we obtain

$$BA = I.$$

$\Rightarrow$ A is nonsingular and $A^{-1} = I + \dfrac{-1}{1 + \langle v, u\rangle} uv^T.$

(2)

- If $\langle v, u \rangle = -1$, then $u = -\frac{v}{\|v\|_2^2}$ (why?)

$\Rightarrow A = I + \vec{u}\vec{v}^T = I - \frac{1}{\|v\|_2^2} v v^T$.

and A is singular since

$$A\vec{v} = v - \frac{1}{\|v\|_2^2} v v^T v = v - v = 0.$$

$\Rightarrow \quad v \in \text{Null}(A)$.

And $\text{Null}(A) \cancel{= sp\{\,\}} = \text{span}\{v\}$. since for any vector $\vec{x} \neq t\vec{v}$ (i.e., x is not in the direction of v).

$$A\vec{x} = \vec{x} - \underbrace{\frac{1}{\|v\|_2^2} \langle \vec{v}, \vec{x} \rangle}\, \vec{v}$$

$$= \vec{x} - s\vec{v} \neq 0.$$

$\Rightarrow \quad \vec{x} \notin \text{Null}(A)$.

4) Let $E = uv^T$. By definition,

$$\|E\|_2 = \sup_{\|x\|_2 = 1} \|Ex\|_2$$

$$= \sup_{\|x\|_2 = 1} \|u v^T x\|_2.$$

$$= \sup_{\|x\|_2 = 1} |\langle v, x \rangle| \, \|u\|_2.$$

By the Cauchy-Schwartz inequality:

$$|\langle v, x \rangle| \leq \|v\|_2 \|x\|_2.$$

$$\therefore \quad \|E\|_2 \leq \sup_{\|x\|_2 = 1} \|v\|_2 \|x\|_2 \|u\|_2 = \|v\|_2 \|u\|_2.$$

③.

$\Rightarrow \quad \|E\|_2 \leq \|v\|_2 \|u\|_2.$

we can achieve "=" by

Taking $x = \dfrac{v}{\|v\|_2}$. then $\|x\|_2 = 1$,

and $\|Ex\|_2 = \|uv^T x\|_2 = \left\|uv^T \dfrac{v}{\|v\|_2}\right\|_2$

$$= \|u\|v\|_2\|_2.$$

$$= \|v\|_2 \|u\|_2.$$

$\therefore \quad \|E\|_2 = \|v\|_2 \|u\|_2.$

It's also true for $\|E\|_F = \|v\|_F \|u\|_F$.

Pf: We observe that

$$E = [v_1\vec{u} \quad v_2\vec{u} \quad \cdots \quad v_n\vec{u}].$$

Then,

$$\|E\|_F^2 = \|v_1\vec{u}\|_2^2 + \|v_2\vec{u}\|_2^2 + \cdots + \|v_n\vec{u}\|_2^2.$$

$$= |v_1|^2 \|u\|_2^2 + |v_2|^2 \|\vec{u}\|_2^2 + \cdots + |v_n|^2 \|\vec{u}\|_2^2$$

$$= (|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2)\|\vec{u}\|_2^2.$$

$$= \|\vec{v}\|_2^2 \|\vec{u}\|_2^2.$$

$$\Rightarrow \quad \|E\|_F = \|\vec{v}\|_2 \|\vec{u}\|_2 = \|\vec{v}\|_F \|\vec{u}\|_F.$$

Note that $\|v\|_2 = \|v\|_F$ and $\|u\|_2 = \|u\|_F$.

④.

5) $A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}$

$\|A\|_1 = \max\{|-2|+|-4|+1,\ |3|+|5|+|-2|,\ |2|+|1|+|4|\}$

$= \max\{7, 10, 7\}$

$= 10$

$\|A\|_\infty = \max\{|-2|+|3|+|2|,\ |-4|+|5|+|1|,\ |1|+|-2|+|4|\}.$

$= \max\{7, 10, 7\}.$

$= 10.$

$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)} \Rightarrow$ ~~will not~~ will learn how to find it later.

$\|A\|_F = \sqrt{|-2|^2 + |3|^2 + |2|^2 + |-4|^2 + |5|^2 + |1|^2 + |1|^2 + |-2|^2 + |4|^2}$

$= 80.$

6) Given $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, show that A^TA is nonsingular if and only if A has full rank.

Pf: ($\Rightarrow$) Suppose A^TA is nonsingular.

If $x \in \mathbb{R}^n$ such that $Ax = 0$, then

$A^TAx = A^T0.$

$A^TAx = 0.$

$\Rightarrow x \in \text{Null}(A^TA).$

Since A^TA is nonsingular, $x = 0.$

$\Rightarrow \text{Null}(A) = \{0\}. \Rightarrow A$ is of full rank.

5

($\Leftarrow$) Suppose A is of full rank.

Let $x \in \text{Null}(A^TA)$, then $A^TAx = 0$.

$\Rightarrow A^Ty = 0$, where $y = Ax$.

$\Rightarrow$ y is orthogonal to columns of A.

But $y \in \text{range}(A)$.

$\Rightarrow$ $y = 0$.

$\Rightarrow$ $Ax = 0$

$\Rightarrow$ $x = 0$.

$\therefore$ $\text{Null}(A^TA) = \{0\}$.

$\therefore$ A^TA is nonsingular.

7) Will learn this week (Week 3).