HW01

Math 152, Winter 2019

Due Week 2

Reference: 1. L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997

Reading: Review what you learned from your Linear Algebra classes. Review vector norms, matrix norms, orthogonality, projections.

1. (a) Let M be the matrix of data points

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}.$$

   What are $M^T M$ and MM^T?

   (b) Prove that if A is any matrix, then $A^T A$ and AA^T are *symmetric*. (Recall that a matrix M is symmetric if $M = M^T$.)

2. (1.4-Trefethen & Bau) Let $f_1, \ldots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \ldots, 8.$$

   (a) Show that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

   (b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?

3. (1.3-Trefethen & Bau) We say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. Show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (Note that the analogous result also holds for lower-triangular matrices.)

4. Recall that a matrix $A \in \mathbb{R}^{m\times n}, m \geq n$, is said to have full rank if its columns are linearly independent, i.e., for $\boldsymbol{a}_j$ the jth column of A, $c_1\boldsymbol{a}_1 + \ldots + c_n\boldsymbol{a}_n = 0 \Longrightarrow c_1 = \ldots = c_n = 0$. Show that A has full rank if and only if no two distinct vectors are mapped to the same vector.

5. Sketch the unit circle $\{\boldsymbol{x}, \|\boldsymbol{x}\|_p = 1\}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ for $p = 1, 2,$ and ∞.

6. (a) Write the definition of the vector norm $\|\boldsymbol{x}\|_2$.

   (b) Show that if Q is an orthogonal matrix, then $\|Q\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$.

   (c) Let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ and

$$Q = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \text{(this is a Hadamard matrix).}$$

   Without calculating $Q\boldsymbol{x}$ directly, what is the value of $\|Q\boldsymbol{x}\|_2$?

7. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^m$, the matrix $A = I + \boldsymbol{u}\boldsymbol{v}^T$ is know as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\boldsymbol{u}\boldsymbol{v}^T$ for some scalar α, and give an expression for α. For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is A singular? If it is singular, what is $\text{Null}(A)$?

8. Given $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$, show that if $E = \boldsymbol{u}\boldsymbol{v}^T$, then $\|E\|_2 = \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|\boldsymbol{u}\|_F\|\boldsymbol{v}\|_F$? Prove it or give a counterexample.

9. Consider the matrix

$$A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}.$$

   What are the ℓ^1, ℓ^2, ℓ^∞-, and Frobenius norms of A?

10. Given $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, show that A^TA is nonsingular if and only if A has full rank.

11. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (a) What is the orthogonal projector P onto $\text{range}(A)$, and what is the image under P of the vector $(1, 2, 3)^T$?

   (b) Same questions for B.