Math 152 - Spring 2019

HW 02 - Solution.

1) We flip a fair coin ten times.

~~a)~~ Define random variables.

$$X_i = \begin{cases} \text{Head} & \text{if we get head in the ith flip, with prob. } 1/2 \\ \text{Tail} & \text{otherwise} \end{cases}$$

a) There are 10 flips of which we choose 5 heads, and there are total of 2^{10} ways to flip the coin. Thus, the probability is

$$\frac{\binom{10}{5}}{2^{10}} = \frac{63}{256}.$$

b) Let $S = X_1 + \dots + X_{10}$ be the number of heads.

$P(\text{more heads than tails})$

$$= P(S=6) + P(S=7) + P(S=8) + P(S=9) + P(S=10)$$

$$= \sum_{k=6}^{10} P(S=k)$$

$$= \sum_{k=6}^{10} \frac{\binom{10}{k}}{2^{10}}$$

$$= \frac{386}{1024}.$$

c) Let $P_4(n)$ be the probability that we have at least 4 consecutive heads after n flips. then

$$P_4(0) = P_4(1) = P_4(2) = P_4(3) = 0.$$

and $P_4(4) = \frac{1}{2^4}.$

For more flips, either we have achieved 4 ~~flips~~ consecutive heads already or we have a string without 4 consecutive heads followed by THHHH. Hence for $n > 4$,

$$P_4(n) = P_4(n-1) + \frac{1}{2^5}\left(1 - P_4(n-5)\right).$$

$$\therefore P_4(10) = P_4(9) + \frac{1}{2^5}\left(1 - P_4(5)\right).$$

$$= P_4(8) + \frac{1}{2^5}\left(1 - P_4(4)\right) + \frac{1}{2^5}\left(1 - P_4(4) - \frac{1}{2^5}\right).$$

$$= P_4(7) + \frac{1}{2^5}\left(1 - P_4(3)\right) + \frac{1}{2^5}\left(1 - P_4(4)\right) + \frac{1}{2^5}\left(1 - P_4(4)\right) - \frac{1}{2^{10}}.$$

$$= \dots = P_4(4) + \frac{1}{2^5}\sum_{k=0}^{4}\left(1 - P_4(k)\right) + \frac{1}{2^5}\left(1 - P_4(4)\right) - \frac{1}{2^{10}}.$$

$$= 2^{-4} + \frac{1}{2^5} \cdot 3(1-0) + \frac{2}{2^5}\left(1 - 2^{-4}\right) - \frac{1}{2^{10}}.$$

2) a) $\mathbb{E}[\max(X_1, X_2)]$.

$$\max(X_1, X_2) = \begin{cases} X_1 & \text{if } X_1 \geq X_2 \\ X_2 & \text{if } X_1 < X_2 \end{cases}.$$

$$\mathbb{E}[\max(X_1, X_2)] = \sum_{j=1}^{k} j \, \mathbb{P}\left(\max(X_1, X_2) = j\right)$$

$\cancel{= \sum_{j=1}^{k} j}$ We need to find $\mathbb{P}(\max(X_1, X_2) = j)$.

$$\mathbb{P}(\max(X_1,X_2)=j) = \mathbb{P}(X_1=j, X_2\leq j) + \mathbb{P}(X_1<j, X_2=j)$$

since X_1 and X_2 are independent

$$= \mathbb{P}(X_1=j)\,\mathbb{P}(X_2\leq j) + \mathbb{P}(X_1<j)\,\mathbb{P}(X_2=j)$$

$$= \frac{1}{k}\;\frac{j}{k} + \frac{j-1}{k}\;\frac{1}{k}$$

$$= \frac{2j-1}{k^2}$$

$$\therefore\ \mathbb{E}[\max(X_1,X_2)] = \sum_{j=1}^{k} j\,\frac{2j-1}{k^2} = \frac{1}{k^2}\sum_{j=1}^{k} 2j^2 - \frac{1}{k^2}\sum_{j=1}^{k} 1$$

$$= \frac{2}{k^2}\sum_{j=1}^{k} j^2 - \frac{1}{k} \qquad \cancel{= \frac{2}{k^2}\,\frac{k(k+1)}{2} - \frac{k}{k^2}}$$

$$= \frac{2}{k^2}\,\frac{k(k+1)(2k+1)}{6} - \frac{1}{k} \qquad \cancel{= k+1-1}$$

$$= \frac{(k+1)(2k+1)}{3k} - \frac{1}{k}$$

b) $$\mathbb{E}[\min(X_1,X_2)] = \sum_{j=1}^{k} j\,\mathbb{P}(\min(X_1,X_2)=j)$$

$$\Rightarrow \mathbb{P}(\min(X_1,X_2)=j) = \mathbb{P}(X_1=j, X_2\geq j) + \mathbb{P}(X_1>j, X_2=j)$$

$$= \mathbb{P}(X_1=j)\,\mathbb{P}(X_2\geq j) + \mathbb{P}(X_1>j)\,\mathbb{P}(X_2=j)$$

$$= \frac{1}{k}\;\frac{k-j+1}{k} + \frac{k-j}{k}\;\frac{1}{k}.$$

$$= \frac{k-j+1+k-j}{k^2}$$

$$= \frac{2(k-j)+1}{k^2}.$$

$$\therefore \mathbb{E}[\min(X_1, X_2)] = \sum_{j=1}^{k} j \, \frac{2(k-j)+1}{k^2}$$

$$= \ldots \text{ (simplify it!)}.$$

Math 152 - HW02.

3) Let $X_i = \begin{cases} 1 & \text{if Alice wins, with probability } 0.7 \\ 0 & \text{if Alice loses, with probability } 0.3 \end{cases}$

(X_i is the random variable indicating whether Alice wins or not in ~~the~~ the ith game).

Let $X = X_1 + X_2 + \dots + X_n$. Thus, X is the random variable indicating the number of games Alice wins in the tournament.

Then, $\mathbb{E}[X_i] = 0.7$ and $\mu = \mathbb{E}[X] = n(0.7)$.

~~App~~ Recall that Alice loses ~~looses~~ ~~lose~~ the tournament if she wins less than half of the games, i.e., $X \leq \frac{n-1}{2}$.

By Chernoff's bound for $\{0,1\}$-valued r.v's. (See other ~~Ch~~ useful forms in the lecture note):

$$\mathbb{P}(X \leq (1-\varepsilon)\mu) \leq e^{-\mu\varepsilon^2/2}.$$

Set $(1-\varepsilon)\mu = \frac{n-1}{2}$, and find ε,

$$0.7n(1-\varepsilon) = 0.5n - 0.5$$

$$0.7n - \varepsilon 0.7n = 0.5n - 0.5$$

$$0.2n + 0.5 = \varepsilon\, 0.7n.$$

$$\frac{2}{7} + \frac{5}{7n} = \varepsilon$$

$$\Rightarrow \quad \varepsilon = \frac{2}{7} + \frac{5}{7n} > \frac{2}{7}. \Rightarrow \varepsilon^2 > \frac{4}{49}.$$

$$\therefore \quad \mathbb{P}\left(X \leq \frac{n-1}{2}\right) = \mathbb{P}(X \leq (1-\varepsilon)\mu) \leq e^{-\mu\varepsilon^2/2} < e^{-0.7n\left(\frac{4}{49}\right)/2}.$$

$$\therefore \quad \mathbb{P}\left(X \leq \frac{n-1}{2}\right) \leq e^{-n/5}.$$

4) Let X be the number of times that a 6 occurs over n throws of the die.

Let $p = \mathbb{P}(X \geq \frac{n}{4})$.

a) Markov's inequality:

$$p = \mathbb{P}(X \geq \tfrac{n}{4}) \leq \frac{\mathbb{E}[X]}{\frac{n}{4}} = \frac{4\mathbb{E}[X]}{n}.$$

We need to evaluate $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\Big] = \cancel{n}\sum_{i=1}^{n} \mathbb{E}[X_i] = \frac{n}{6}.$$

where X_i is the result we get at the ith throws, i.e. $X_i = \begin{cases} 1 & \text{w/ pro } 1/6 \\ 0 & \text{otherw w/ prob } 5/6 \end{cases}$

$$\therefore \boxed{p \leq \frac{4\left(\frac{n}{6}\right)}{n} = \frac{4}{6} = \frac{2}{3}}.$$

b) Chebyshev's inequality:

$$\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \frac{Var[X]}{t^2}$$

$$Var[X] = Var[\sum_i X_i] = \sum_{i=1}^{n} Var[X_i] = n \cdot \frac{5}{36} = \frac{5n}{36}.$$

$$\therefore \mathbb{P}(X \geq \tfrac{n}{4}) = \mathbb{P}(X - \tfrac{n}{6} \geq \tfrac{n}{4} - \tfrac{n}{6})$$

$$= \mathbb{P}(X - \mathbb{E}[X] \geq \tfrac{n}{12})$$

$$\leq \mathbb{P}(|X - \mathbb{E}[X]| \geq \tfrac{n}{12})$$

Chebyshev's ineq. with $t = \frac{n}{12}$

$$\leq \frac{Var[X]}{(n/12)^2}$$

$$= \frac{12^2}{n^2} \cdot \frac{5n}{36}$$

$$= \frac{\cancel{1}20}{n}.$$

$$\therefore \boxed{p \leq \frac{\cancel{4}20}{n}}$$

c) Chernoff's inequality:

$$\cancel{p = \mathbb{P}\left(X \geq \frac{n}{4}\right)}$$

$$\mathbb{P}\left(X \geq (1+\varepsilon)\mu\right) \leq e^{-\mu\varepsilon^2/4}.$$

We want $(1+\varepsilon)\mu = \frac{n}{4} \Rightarrow (1+\varepsilon)\frac{n}{6} = \frac{n}{4}$

$$1+\varepsilon = \frac{3}{2} \Rightarrow \varepsilon = \frac{1}{2}.$$

$$\therefore\ p = \mathbb{P}\left(X \geq \frac{n}{4}\right) = \mathbb{P}\left(X \geq (1+\varepsilon)\mu\right) \leq e^{-\frac{n}{6}\cdot\left(\frac{1}{2}\right)^2/4} = e^{-n/96}$$

$$\therefore\ \boxed{p \leq e^{-n/96}}$$

5) $X_i = \begin{cases} 1 & \text{w/p } 1/2 \\ -1 & \text{w/p } 1/2 \end{cases}$

Let $S = \sum_{i=1}^{n} X_i$.

a) Since $Y = |S|$, Y is a non-negative random variable, Markov's inequality holds for for Y. That is, for $t > 0$,

$$\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}.$$

b) Chebyshev's inequality: for S

$$\mathbb{P}\left(|S - \mathbb{E}[S]| \geq t\right) \leq \frac{\mathrm{Var}[S]}{t^2}.$$

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}[X_i] = 0.$$

$$\mathrm{Var}[S] = \mathrm{Var}\left[\sum_i X_i\right] = \sum_i \mathrm{Var}[X_i] = n$$

$$\therefore\ \mathbb{P}(|S| \geq t) \leq \frac{n}{t^2}.$$

c) (This is similar to the way we prove Chernoff's bound).

$$\mathbb{P}(S \geq a) = \mathbb{P}(X_1 + \cdots + X_n \geq a)$$

$$= \mathbb{P}(\lambda \sum_i X_i \geq \lambda a) \quad \text{for } \lambda \geq 0$$

$$= \mathbb{P}(e^{\lambda \sum_i X_i} \geq e^{\lambda a}) \quad \text{since } e^x \text{ is monotone.}$$

$$\text{Markov's} \leftarrow \leq e^{-\lambda a}\, \mathbb{E}[e^{\lambda \sum X_i}]$$

Now, let's consider $\mathbb{E}[e^{\lambda \sum_i X_i}]$.

$$\mathbb{E}[e^{\lambda \sum_i X_i}] = \mathbb{E}[e^{\lambda X_1} \cdot e^{\lambda X_2} \cdots e^{\lambda X_n}].$$

$$= \prod_i \mathbb{E}[e^{\lambda X_i}] \quad \text{since } X_i\text{'s are independent.}$$

$$= \prod_i \left(e^{\lambda} \cdot \mathbb{P}(X_i = 1) + e^{-\lambda}\, \mathbb{P}(X_i = -1)\right).$$

$$= \prod_i \frac{e^{\lambda} + e^{-\lambda}}{2}.$$

$$= \prod_i \cosh(\lambda)$$

$$= (\cosh(\lambda))^n.$$

Consider $\lambda \in [0, 1]$, then $\cosh(\lambda) \leq e^{\lambda^2/2}$.

$$\therefore (\cosh(\lambda))^n \leq e^{\lambda^2 n/2}$$

$$\therefore \mathbb{P}(S \geq a) \leq e^{-\lambda a}\, \mathbb{E}[e^{\lambda \sum_i X_i}] = e^{\frac{\lambda^2 n}{2} - \lambda a}$$

Recall that min of $\frac{\lambda^2 n}{2} - \lambda a$ occurs at $\lambda = \cancel{\frac{2a}{n}}\ \frac{a}{n}$. ← take this one

$$\therefore \mathbb{P}(S \geq a) \leq e^{\frac{a^2}{n^2} \cdot \frac{n}{2} - \frac{a^2}{n}} = e^{-a^2/2n}.$$