HW02

Due Week 3-4

References: Foundations of Data Science by Blum, Hopcroft, and Kannan [BHK]; Probability and Computing by Mitzenmacher and Upfal [MU]

Reading: Sections 12.5, 12.6, 12.7 in [BHK] and lecture notes. Sections 1.2, 2.1, 2.2, 3.1, 3.2, 3.3, and Chapter 4 in [MU]. Note that Chernoff's bound in the lecture note may be slightly different from Chapter 4 in [MU], but don't worry too much about it. Just read the text to get some ideas and use the one from the lecture note (I simplify it so that it's easier for you to use). Review (discrete and continuous) random variables, sample space, events, independent and dependent r.v.'s, probabilistic inequalities.

1. We flip a fair coin ten times. Find the probability of the following events.
   (a) The number of heads and the number of tails are equal.
   (b) There are more heads than tails.
   (c) We flip at least four consecutive heads.

2. Suppose that we roll twice a fair k-sided die with the numbers 1 through k on the die's faces, obtaining values X_1 and X_2. What are $\mathbb{E}[\max(X_1, X_2)]$ and $\mathbb{E}[\min(X_1, X_2)]$?

3. Alice and Bob play checkers often. Alice is a better player, so the probability that she wins any given game is 0.7, independent of all other games. They decide to play a tournament of n games. Bound the probability that Alice loses the tournament using a Chernoff bound.

4. We have a standard six-sided die. Let X be the number of times that a 6 occurs over n throws of the die. Let p be the probability of the event $X \geq n/4$. Compare the best upper bound on p that you can obtain using Markov's inequality, Chebyshev's inequality, and Chernoff bounds.

5. Let $X_1, \ldots, X_n$ be independent $\{-1, 1\}$-valued random variables. Each X_i takes the value 1 with probability 1/2 and else -1. Let $S = \sum_{i=1}^n X_i$.
   (a) Let Y be a random variable defined as $Y = |S|$. Prove that Markov's inequality holds for Y.
   (b) Prove Chebyshev's inequality for the above random variable S.
   (c) Show that for any $a > 0$,

$$\mathbb{P}(S \geq a) \leq 2e^{-a^2/2n}.$$

   Hint: For $t > 0$, show that $\mathbb{E}[e^{tX_i}] = \cosh(tX_i)$. Then, use the fact that $\cosh(x) \leq e^{x^2/2}$ for $x \in [0, 1]$.