Math 152 - Spring 2019

Homework 1 Solution

2) a) Let $M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}$. Then

$$MM^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix}$$

and $M^TM = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$.

b) $(A^TA)^T = A^T(A^T)^T = A^TA$.

and $(AA^T)^T = (A^T)^T A^T = AA^T$.

$\therefore$ A^TA and AA^T are symmetric.

3) Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$.

Show that A full rank if and only if given any $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{y}$, then $A\vec{x} \neq A\vec{y}$.

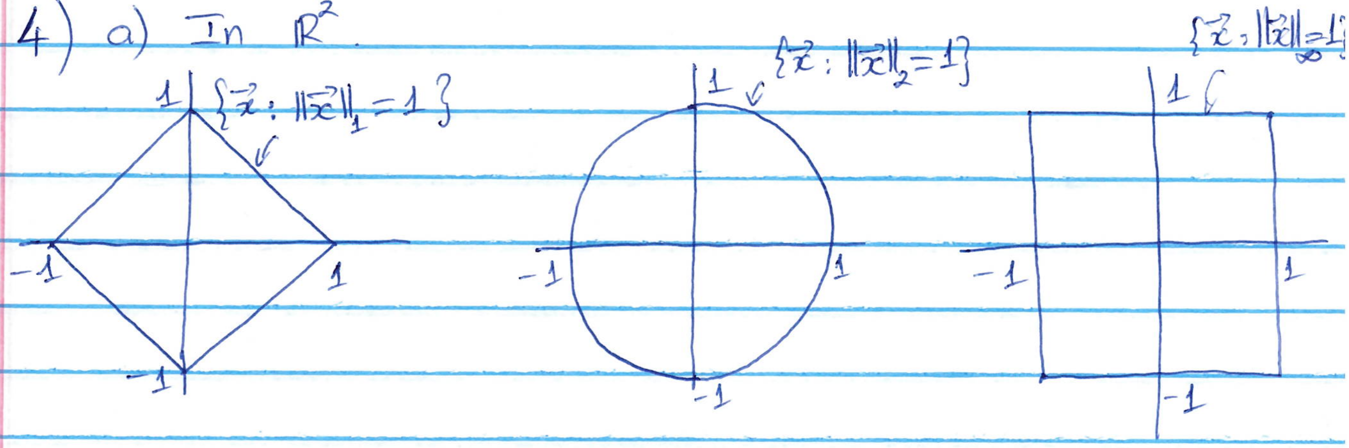
Proof: ($\Rightarrow$) Suppose that A is of full rank.

Then $\text{Null}(A) = \{\vec{0}\}$.

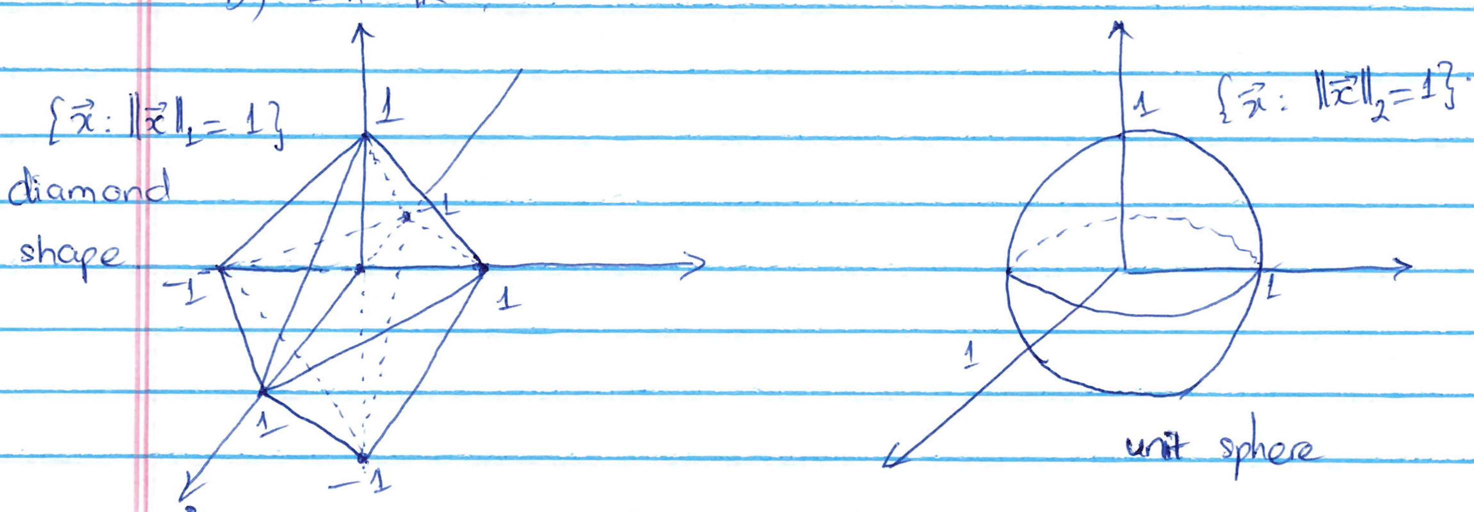
Take any vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ such that
$\vec{x} \neq \vec{y}$, i.e., $\vec{x} - \vec{y} \neq \vec{0}$
Then $\vec{x} - \vec{y} \notin \mathrm{Nul}(A)$
and $A(\vec{x} - \vec{y}) \neq \vec{0}$.
$\therefore$ $A\vec{x} \neq A\vec{y}$.

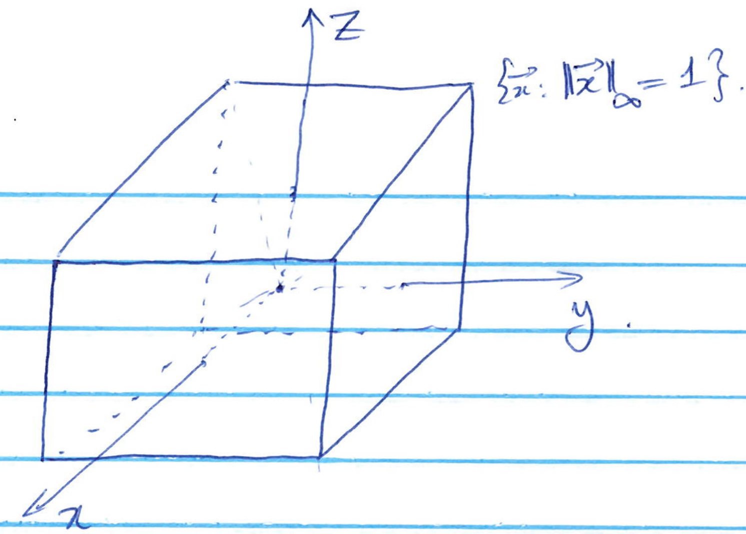
($\Leftarrow$) Take any $\vec{x} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{0}$, then
$$A\vec{x} \neq A\vec{0} = \vec{0}.$$
$$\Rightarrow \quad \vec{x} \notin \mathrm{Nul}(A).$$
$\therefore$ $\mathrm{Nul}(A) = \{\vec{0}\}$
$\therefore$ A is of full rank.

4) a) In $\mathbb{R}^2$.



b) In $\mathbb{R}^3$.





5) a) $\|\vec{x}\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}$ for $\vec{x} \in \mathbb{R}^n$.

b) Suppose that Q is orthogonal, i.e., $Q^T = Q^{-1}$.
Then

$$\begin{aligned}\|Q\vec{x}\|_2^2 &= \langle Q\vec{x}, Q\vec{x}\rangle \\ &= (Q\vec{x})^T Q\vec{x} \\ &= \vec{x}^T \underbrace{Q^T Q}\vec{x} \\ &= \vec{x}^T I \vec{x} \\ &= \vec{x}^T \vec{x} \\ &= \|\vec{x}\|_2^2.\end{aligned}$$

$\therefore \|Q\vec{x}\|_2 = \|\vec{x}\|_2$.

c) It's easy to show that $Q^T Q = I$.
$\Rightarrow$ Q is an orthogonal matrix.
From part b),

$$\|Qx\|_2 = \|x\|_2 = \sqrt{10}.$$

6) For $\vec{u} \neq 0$ and $\vec{v} \neq 0$ in $\mathbb{R}^m$, let $A = I + \vec{u}\vec{v}^T$.

~~Suppose A is nonsingular.~~

Consider $B = I + \alpha \vec{u}\vec{v}^T$. for some scalar α. Then

$$BA = (I + \alpha\vec{u}\vec{v}^T)(I + \vec{u}\vec{v}^T)$$

$$= I + \vec{u}\vec{v}^T + \alpha\vec{u}\vec{v}^T + \alpha\vec{u}\underbrace{(\vec{v}^T\vec{u})}_{\langle\vec{v},\vec{u}\rangle}\vec{v}^T.$$

$$= I + (1 + \alpha + \alpha\langle\vec{v},\vec{u}\rangle)\vec{u}\vec{v}^T.$$

- If $\langle\vec{v},\vec{u}\rangle \neq -1$, take $\alpha = \dfrac{-1}{1+\langle\vec{v},\vec{u}\rangle}$, we obtain

$$BA = I.$$

$\therefore$ A is nonsingular and $A^{-1} = B = I - \dfrac{1}{1+\langle\vec{v},\vec{u}\rangle}\vec{u}\vec{v}^T.$

- If $\langle\vec{v},\vec{u}\rangle = -1$, then

$$A\vec{u} = (I + \vec{u}\vec{v}^T)\vec{u} = \vec{u} + \vec{u}\vec{v}^T\vec{u}.$$

$$= \vec{u} + \vec{u}\langle\vec{v},\vec{u}\rangle.$$

$$= \vec{u} - \vec{u}.$$

$$= \vec{0}.$$

$\therefore$ $\vec{u} \in Nul(A)$.

$\therefore$ A is singular.

We will show that $Nul(A) = span\{\vec{u}\}$.

1) Let $\vec{x} \in Nul(A)$, then

$$A\vec{x} = 0$$

$$(I + \vec{u}\vec{v}^T)\vec{x} = 0.$$

$$\vec{x} + \vec{u}\vec{v}^T\vec{x} = 0.$$

$$\vec{x} = -\langle\vec{v},\vec{x}\rangle\vec{u}.$$

$\therefore$ $\vec{x} \in span\{\vec{u}\}$.

$\therefore$ $Nul(A) \subset span\{\vec{u}\}$.

ii) let $\vec{x} \in \text{span}\{\vec{u}\}$. Then $\exists\, t \in \mathbb{R}$ such that $\vec{x} = t\vec{u}$.

and
$$A\vec{x} = (I + \vec{u}\vec{v}^T)\vec{x}$$
$$= \vec{x} + \vec{u}\langle \vec{v}, \vec{x} \rangle$$
$$= t\vec{u} + \vec{u}\langle \vec{v}, t\vec{u} \rangle$$
$$= t\vec{u} + t\vec{u}\langle \vec{v}, \vec{u} \rangle$$
$$= t\vec{u} - t\vec{u}$$
$$= 0.$$

$\therefore \quad \vec{x} \in Nul(A)$.

7) a) Let $E = uv^T$. By definition,
$$\|E\|_2 = \sup_{\|x\|_2 = 1} \|Ex\|_2$$
$$= \sup_{\|x\|_2 = 1} \|u \underbrace{v^T x}_{\langle v, x \rangle}\|_2$$
$$= \sup_{\|x\|_2 = 1} |\langle v, x \rangle| \, \|u\|_2$$

Cauchy-Schwartz $|\langle v, x \rangle| \le \|v\|_2 \|x\|_2$
$$\le \sup_{\|x\|_2 = 1} \|v\|_2 \|x\|_2 \|u\|_2.$$

$$\therefore \quad \|E\|_2 \le \|v\|_2 \|u\|_2.$$

We can achieve "=" by taking $x = \frac{v}{\|v\|_2}$.

Then $\|x\|_2 = 1$ and
$$\|Ex\|_2 = \|uv^T x\|_2 = \left\| uv^T \frac{v}{\|v\|_2} \right\|_2$$
$$= \left\| u \frac{\|v\|_2^2}{\|v\|_2} \right\|_2$$
$$= \|u\|_2 \|v\|_2.$$

It's also true that $\|E\|_F = \|u\|_F \, \|v\|_F$.

Since $E = uv^T = [v_1\vec{u} \quad v_2\vec{u} \quad \dots \quad v_n\vec{u}]$.

then, $\|E\|_F^2 = \|v_1\vec{u}\|_2^2 + \|v_2\vec{u}\|_2^2 + \dots + \|v_n\vec{u}\|_2^2$

$= |v_1|^2 \|\vec{u}\|_2^2 + |v_2|^2 \|\vec{u}\|_2^2 + \dots + |v_n| \, \|\vec{u}\|_2^2$.

$= (|v_1|^2 + |v_2|^2 + \dots + |v_n|^2) \|\vec{u}\|_2^2$

$= \|v\|_2^2 \quad \|\vec{u}\|_2^2$.

$\therefore \quad \|E\|_F = \|v\|_2 \, \|u\|_2 = \|v\|_F \, \|u\|_F$.

8) $A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}$

a) $\|A\|_F = \sqrt{|-2|^2 + |3|^2 + |2|^2 + |-4|^2 + |5|^2 + 1^2 + 1^2 + |-2|^2 + 1^2}$

$= 80$.

b) $\|A\|_1 = \max\{|-2| + |-4| + |1|, \ |3| + |5| + |-2|, \ |2| + |1| + |4|\}$

$= \max\{7, \quad 10, \quad 7\}$

$= 10$.

c) $\|A\|_\infty = \max\{|-2| + |3| + |2|, \ |-4| + |5| + |1|, \ |1| + |-2| + |4|\}$

d) $\|A\|_2 =$??? Will learn how to find it later.

9) $A \in \mathbb{R}^{m \times n}$ with $m \geq n$,

A^TA is nonsingular if and only if A is of full rank

proof: ($\Rightarrow$) Suppose that A^TA is nonsingular.

If $\vec{x} \in$ Nul(A), then

$\vec{x} \neq \vec{0}$. $\quad A\vec{x} = 0$

$A^TAx = A^T0 = 0$.

$\therefore \quad x \in Nul(A^TA)$

Since A^TA is nonsingular, $x = 0$.

$\therefore \quad Nul(A) = \{\vec{0}\}$

$\Rightarrow$ A is of full rank.

$(\Leftarrow)$ Suppose A is of full rank.

Let $\underset{\vec{0}}{\overset{}{x}} \underset{\#}{} \in Null(A^TA)$, then $A^TA\vec{x} = 0$

$\therefore \quad A^Ty = 0$, where $y = Ax$

$\therefore$ y is orthogonal to columns of A,

But $y \in Col(A)$.

$\therefore \quad y = 0$

$Ax = 0$.

$x = 0$ since A is of full rank.

$\therefore \quad Null(A^TA) = \{\vec{0}\}$.

$\therefore \quad A^TA$ is nonsingular.

10) We need to find a vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ such that $\|x\|_2 = 1$ which achieves the maximum ℓ^1-norm.

First, let's prove the following inequality:

For $\vec{x} \in \mathbb{R}^2$, $\|\vec{x}\|_1 \leq \sqrt{2}\, \|\vec{x}\|_2$.

proof: $\|\vec{x}\|_1 = |x_1| + |x_2|$.

$= (1, 1) \cdot (|x_1|, |x_2|)$.

dot product of two vectors $(1,1)$ and $(|x_1|, |x_2|)$.

Cauchy-Schwartz $\leq \|(1, 1)\|_2 \;\|(|x_1|, |x_2|)\|_2$

$= \sqrt{2} \quad \|\vec{x}\|_2$.

"=" occurs when $|x_1| = |x_2|$.

Thus, for any $\vec{x} \in \mathbb{R}^2$ such that $\|\vec{x}\|_2 = 1$,

$$\|\vec{x}\|_1 \leq \sqrt{2}.$$

and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

are points with unit ℓ^2-norm and maximum ℓ^1-norm.

11) $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$.

a) By definition,

$$\|AB\|_2 = \sup_{\|x\|_2 = 1} \|ABx\|_2$$

$$\leq \sup_{\|x\|_2 = 1} \|A\|_2 \|Bx\|_2.$$

$$= \|A\|_2 \sup_{\|x\|_2 = 1} \|Bx\|_2$$

$$= \|A\|_2 \|B\|_2.$$

b) Suppose $B = [\vec{b}_1 \ \dots \ \vec{b}_n]$. then

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_n].$$

$$(*) \quad \|AB\|_F^2 = \|Ab_1\|_2^2 + \|Ab_2\|_2^2 + \dots + \|A\vec{b}_n\|_2^2.$$

Suppose $A = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_p & - \end{bmatrix}$ where A_j is j-th row of A.

then,

$$Ab_k = \begin{bmatrix} \langle A_1, b_k \rangle \\ \vdots \\ \langle A_p, b_k \rangle \end{bmatrix}$$

$$\|Ab_k\|_2^2 = \sum_{j=1}^{p} |\langle A_j, b_k \rangle|^2 \underset{\text{Cauchy-Schwarz}}{\leq} \sum_{j=1}^{p} \|A_j\|_2^2 \|b_k\|_2^2.$$

$$= \left(\sum_{j=1}^{p} \|A_j\|_2^2\right) \|b_k\|_2^2.$$

$$= \|A\|_F^2 \|b_k\|_2^2.$$

Therefore, each term in (*) can be bounded by

$$(*) \leq \|A\|_F^2 \|b_1\|_2^2 + \|A\|_F^2 \|b_2\|_2^2 + \dots + \|A\|_F^2 \|b_n\|_2^2.$$

$$= \|A\|_F^2 \left(\|b_1\|_2^2 + \|b_2\|_2^2 + \dots + \|b_n\|_2^2 \right).$$

$$= \|A\|_F^2 \|B\|_F^2.$$