FinalvA, Math 152, Winter 2019 — Total Points: 50 (+ 3)

**Name:** Key **PID:**

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- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

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**DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO**

(This exam is worth 50 points + 3 bonus points)

**Problem 0.**(2 point.) Follows the instructions on this exam and any additional instructions given during the exam.

**Name:**
**Problem 1.**(6 points.)

a) (3 points) Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

Show that $AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}$.

b) (3 points) Express $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$ as a product of two matrices.

a) $\cancel{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [b_{11}\ b_{12}\ b_{13}] + \begin{bmatrix} 0 \\ 3 \end{bmatrix} [b_{21}\ b_{22}\ b_{23}] + \begin{bmatrix} 5 \\ 4 \end{bmatrix} [b_{31}\ b_{32}\ b_{33}]$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 2b_{11} & 2b_{12} & 2b_{13} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 3b_{21} & 3b_{22} & 3b_{23} \end{bmatrix} + \begin{bmatrix} 5b_{31} & 5b_{32} & 5b_{33} \\ 4b_{31} & 4b_{32} & 4b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} + 5b_{31} & b_{12} + 5b_{32} & b_{13} + 5b_{33} \\ 2b_{11} + 3b_{21} + 4b_{31} & 2b_{12} + 3b_{22} + 4b_{33} & 2b_{13} + 3b_{23} + 4b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = AB.$$

b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [0\ \ 4] + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} [2\ \ 1] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$.

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Ver B: a) similar way to prove.

b) $\begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 & 1 \\ -1 & 1 \end{bmatrix}$.

**Name:**

**Problem 2.**(6 points.) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let $A_{i,j}$ be the $2 \times 2$ matrix whose entries are all zeros except the $(i,j)$ entry which is set to $a_{i,j}$, the $(i,j)$ entry of $A$.

a) (2 points) Let $|A|_1 = \sum_{i,j} |a_{i,j}|$. Find $|A|_1$.

b) (2 points) Let $p_{i,j} = \frac{|a_{i,j}|}{|A|_1}$. Find $\frac{1}{p_{1,2}^2} A_{1,2} A_{1,2}^T$.

c) (2 points) Let $B$ be a matrix valued random variable defined by $B = \frac{1}{p_{i,j}} A_{i,j}$ with probability $p_{i,j}$. Show that $\mathbb{E}[BB^T] = 10 \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$.

a) $|A|_1 = 1 + 2 + 3 + 4 = 10.$

b) $P_{1,2} = \frac{2}{10} \Rightarrow \frac{1}{P_{1,2}^2} A_{12} A_{1,2}^T = \frac{10^2}{2^2} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$

$= 10^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

c) $\mathbb{E}[BB^T] = \sum_{i=1}^{2} \sum_{j=1}^{2} P_{i,j} \frac{1}{P_{i,j}^2} A_{i,j} A_{i,j}^T.$

$= \cancel{\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{1}{P_{i,j}} A_{i,j} A_{i,j}^T}.$

$= \sum_{i=1}^{2} \sum_{j=1}^{2} P_{ij} \frac{|A_1|^2}{|a_{i,j}|^2} \underbrace{A_{i,j} A_{i,j}^T}.$

$= \sum_{j=1}^{2} P_{1j} \frac{|A_1|^2}{|a_{1j}|^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{j=1}^{2} P_{2j} \frac{|A_1|^2}{|a_{2j}|^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

$= |A|_1 \sum_{j=1}^{2} a_{1j} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + |A_1| \sum_{j=1}^{2} a_{2j} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

$= 10 \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}.$

Ver B:

b) $P_{2,1} = \frac{3}{10}.$

$\Rightarrow \frac{1}{P_{2,1}^2} A_{2,1} A_{2,1}^T = 10^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

**Name:**

**Problem 3.**(6 points.) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Define a matrix valued random variable $X$ by $X = 3A(:,k)B(k,:)$ with probability $1/3$ for $k = 1, 2, 3$. Here, $A(:,k)$ and $B(k,:)$ denote the $k$th column of $A$ and the $k$th row of $B$, respectively.

a) (2 points) Calculate $\|A\|_F^2 \|B\|_F^2$.

b) (2 points) Calculate $\mathbb{E}[X]$.

c) (2 points) Calculate $\mathrm{Var}[X]$.

a) $\|A\|_F^2 \|B\|_F^2 \; (4)(7) = 28.$

b) $\mathbb{E}[X] = AB = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$

c) $\mathrm{Var}[X] = \sum_{k=1}^{3} 3 \|A(:,k)\|^2 \|B(k,:)\|^2$

$= 3 \left[ (2)(3) + 1(2) + 1(2) \right].$

$= 3(10) = 30.$

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Ver B: a) $\|A\|_F^2 \; \|B\|_F^2 = 28.$

b) $\mathbb{E}[X] = AB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$

c) $\mathrm{Var}[X] = 3(2 \cdot 3 + 1 \cdot 2 + 1 \cdot 2) = 30.$

**Name:**

**Problem 4.**(6 points.) Let $A$ be an $m \times n$ matrix and $B$ an $n \times p$ matrix.

a) (3 points) What is the running time to compute $AB$?

b) (3 points) Define a matrix valued random variable $X$ by $X = \frac{1}{p_k} A(:,k)B(k,:)$ with probability $p_k$, where $A(:,k)$ is the $k$th column of $A$, $B(k,:)$ is the $k$th row of $B$, and $p_k = \frac{\|A(:,k)\|_2^2}{\|A\|_F^2}$. We know that $\mathbb{E}[X] = AB$ and $\mathrm{Var}[X] = \mathbb{E}[\|X - AB\|_F^2] \leq \|A\|_F^2 \|B\|_F^2$. How can you improve the result? That is, can you find a matrix valued random variable $Z$ such that $\mathbb{E}[Z] = AB$ and $Z$ has a smaller variance than $X$?

a) $O(mnp)$.

b) $Z = \frac{X_1 + \dots + X_s}{s}$ where $X_1, \dots, X_s$ independent copies of $X$.

$\Rightarrow \mathbb{E}[Z] = \mathbb{E}[X] = AB$.

$\mathrm{Var}[Z] = \frac{\mathrm{Var}[X]}{s} \leq \cancel{\frac{\mathrm{Var}[X]}{s}}\ \mathrm{Var}[X]$.

**Name:**

**Problem 5.**(6 points.) Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, show that if $E = \vec{u}\vec{v}^T$ (E is an $n \times n$ matrix), then $\|E\|_2 = \|\vec{u}\|_2\|\vec{v}\|_2$. (Hint: use SVD)

$$E = uv^T = \underbrace{\frac{u}{\|u\|_2}}_{U} \underbrace{[\|u\|_2 \|v\|_2]}_{\Sigma} \underbrace{\frac{v^T}{\|v\|_2}}_{V^T}.$$

$$\Rightarrow \quad \sigma_1 = \|u\|_2 \|v\|_2 . = \|E\|_2 .$$

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Ver B: same!

**Name:**

**Problem 6.**(6 points.) We have a standard six-sided die. Let $X$ be the number of times that a 6 occurs over $n$ throws of the die. Let $p = \mathbb{P}(X \geq \frac{n}{3})$.

a) (3 points) Use Markov's inequality to bound $p$.

b) (3 points) Use Chebyshev's inequality to bound $p$.

$$X_i = \begin{cases} 1 & \text{if } 6 \text{ at } i\text{th throw. with prob. } 1/6 \\ 0 & \text{otw} \quad 5/6 \end{cases}$$

$$X = X_1 + \dots + X_n.$$

$$\mathbb{E}[X] = \frac{n}{6}.$$

$$\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2.$$

$$= \sum_{i=1}^{n} \left(\frac{1}{6} - \frac{1}{36}\right) = \frac{5n}{36}.$$

a) $\mathbb{P}(X \geq \frac{n}{3}) \leq \frac{\mathbb{E}[X]}{\frac{n}{3}} = \frac{n/6}{n/3} = \frac{1}{2}.$

b) $\mathbb{P}(X \geq \frac{n}{3}) = \mathbb{P}(X - \frac{n}{6} \geq \frac{n}{6}) \leq \frac{\mathrm{Var}[X]}{(\frac{n}{6})^2} = \frac{\frac{5n}{36}}{(\frac{n}{6})^2} = \frac{5}{n}.$

Ver B: a) $\mathbb{P}(X \geq \frac{n}{2}) \leq \frac{n/6}{n/2} = \frac{1}{3}.$

b) $\mathbb{P}(X \geq \frac{n}{2}) = \mathbb{P}(X - \frac{n}{6} \geq \frac{n}{2} - \frac{n}{6})$

$$\leq \frac{5n/36}{(\frac{n}{3})^2} = \frac{45}{36n} = \frac{5}{4n}.$$

**Name:**

**Problem 7.**(6 points.) Let $\sum_{i=1}^{r} \sigma_i u_i v_i^T$ be the SVD of a matrix $A$. Show that $\|u_2^T A\|_2 = \sigma_2$.

$$u_2^T A = u_2^T \sum_{i=1}^{r} \sigma_i u_i v_i^T = \sum_{i=1}^{r} \sigma_i u_2^T u_i v_i^T$$

$$= \sigma_2 u_2^T u_2 v_2^T \qquad \text{since } u_2^T u_i = 0 \text{ if } i \neq 2.$$

$$= \sigma_2 v_2^T.$$

$$\Rightarrow \|u_2^T A\|_2 = \|\sigma_2 v_2^T\|_2 = \sigma_2.$$

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Ver B: $\quad u_1^T A = \sigma_1 v_1$

$$\|u_1^T A\|_2 = \|\sigma_1 v_1\|_2 = \sigma_1.$$

**Name:**

**Problem 8.**(6 points.) Let $X$ be any random variable with mean 0 and variance 1.

a) (2 points) Use Chebyshev's inequality to bound $\mathbb{P}(|X - \mathbb{E}[X]| \geq 0.3)$.

b) (2 points) Explain why the bound you get from Part a) is not useful?

c) (2 points) Let's consider $n$ independent copies $X_1, \ldots, X_n$ of $X$. Let $Z = \frac{X_1+\ldots+X_n}{n}$. Use Chebyshev's inequality to bound $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq 0.3)$.

~~a) $\mathbb{P}(|X - \mathbb{E}[X]| \geq 0.3) \leq \frac{\mathbb{E}[|X|]}{0.3}$~~ ~~since we cannot find~~ $\checkmark$. $\mathbb{E}[|X|]$.

~~It's ok if students~~ leave this as an answer.

or ~~if they write $\mathbb{E}[|X|]=0$~~

a) $\mathbb{P}(|X - \mathbb{E}[X]| \geq 0.3) \leq \frac{Var[X]}{0.3^2} = \frac{1}{(\frac{3}{10})^2} = \frac{100}{9}$

b) Part a) is not useful because $\frac{100}{9} > 1$.

c) $\mathbb{E}[Z] = 0 \qquad Var[Z] = \frac{1}{n}$.

$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq 0.3) \leq \frac{1/n}{0.3^2} = \frac{1}{0.3^2 n}$.

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Ver B: a) $\mathbb{P}(| \quad | \geq 0.2) \leq \frac{1}{(\frac{2}{10})^2} = \frac{100}{4}$.

b) ....

c) $\mathbb{P}( \quad \geq 0.2) \leq \frac{1}{0.2^2 n}$.

**Name:**

**Problem 9.**(3 points.) Tell me an application of data stream that you know.