Name: **PID:**

- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 50 points + 3 bonus points)

Problem 0.(2 point.) Follows the instructions on this exam and any additional instructions given during the exam.

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Problem 1.(6 points.)

a) (3 points) Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

Show that $AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}$.

b) (3 points) Express $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$ as a product of two matrices.

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Problem 2.(6 points.) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let $A_{i,j}$ be the 2×2 matrix whose entries are all zeros except the (i, j) entry which is set to $a_{i,j}$, the (i, j) entry of A.

a) (2 points) Let $|A|_1 = \sum_{i,j} |a_{i,j}|$. Find $|A|_1$.

b) (2 points) Let $p_{i,j} = \frac{|a_{i,j}|}{|A|_1}$. Find $\frac{1}{p_{1,2}^2} A_{1,2} A_{1,2}^T$.

c) (2 points) Let B be a matrix valued random variable defined by $B = \frac{1}{p_{i,j}} A_{i,j}$ with probability $p_{i,j}$. Show that $\mathbb{E}[BB^T] = 10^2 \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$.

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Problem 3.(6 points.) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Define a matrix valued random variable X by $X = 3A(:,k)B(k,:)$ with probability $1/3$ for $k = 1, 2, 3$. Here, $A(:,k)$ and $B(k,:)$ denote the kth column of A and the kth row of B, respectively.

a) (2 points) Calculate $\|A\|_F^2 \|B\|_F^2$.

b) (2 points) Calculate $\mathbb{E}[X]$.

c) (2 points) Calculate $\mathrm{Var}[X]$.

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Problem 4.(6 points.) Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

a) (3 points) What is the running time to compute AB?

b) (3 points) Define a matrix valued random variable X by $X = \frac{1}{p_k} A(:,k)B(k,:)$ with probability p_k, where $A(:,k)$ is the kth column of A, $B(k,:)$ is the kth row of B, and $p_k = \frac{\|A(:,k)\|_2^2}{\|A\|_F^2}$. We know that $\mathbb{E}[X] = AB$ and $\mathrm{Var}[X] = \mathbb{E}[\|X - AB\|_F^2] \leq \|A\|_F^2 \|B\|_F^2$. How can you improve the result? That is, can you find a matrix valued random variable Z such that $\mathbb{E}[Z] = AB$ and Z has a smaller variance than X?

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Problem 5.(6 points.) Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, show that if $E = \vec{u}\vec{v}^T$ (E is an $n \times n$ matrix), then $\|E\|_2 = \|\vec{u}\|_2\|\vec{v}\|_2$. (Hint: use SVD)

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Problem 6.(6 points.) We have a standard six-sided die. Let X be the number of times that a 6 occurs over n throws of the die. Let $p = \mathbb{P}(X \geq \frac{n}{3})$.

a) (3 points) Use Markov's inequality to bound p.

b) (3 points) Use Chebyshev's inequality to bound p.

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Problem 7.(6 points.) Let $\sum_{i=1}^{r} \sigma_i u_i v_i^T$ be the SVD of a matrix A. Show that $\|u_2^T A\|_2 = \sigma_2$.

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Problem 8.(6 points.) Let X be any random variable with mean 0 and variance 1.

a) (2 points) Use Chebyshev's inequality to bound $\mathbb{P}(|X - \mathbb{E}[X]| \geq 0.3)$.

b) (2 points) Explain why the bound you get from Part a) is not useful?

c) (2 points) Let's consider n independent copies $X_1, \ldots, X_n$ of X. Let $Z = \frac{X_1 + \ldots + X_n}{n}$. Use Chebyshev's inequality to bound $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq 0.3)$.

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Problem 9.(3 points.) Tell me an application of data stream that you know.